February 17, 2016

Name:

You have **80** minutes to complete the test. Each problem is worth **20** points. You cannot use any books or notes.

1. Label each of the following statements as either true of false.
   a. $A \cup \emptyset = A$ is true for any set A. True
   b. If $A \cup X = A \cup Y$ holds only for $X = Y$. False
   c. The empty set $\emptyset$ has no subsets. False
   d. Let A and B be sets. If $\mathsf{P}(A) = \mathsf{P}(B)$ only if $A = B$. True
   e. The union of two subsets A and B of the set X is the largest subset of X that contains A and B. False
   f. The intersection of two subsets A and B of the set X is the largest subset of X that is contained in A and B. True
   g. $x_1 = x_2 \to f(x_1) = f(x_2)$ defines injectivity of f. False
   h. $f(x_1) \neq f(x_2) \to x_1 \neq x_2$ defines injectivity of f. False
   i. Let $f : A \to B$ be any map. If there is a map $g : B \to A$ such that $f \circ g = id_B$ then f must be surjective. True
   j. If every injective map $f : A \to A$ is also surjective then A must be finite. True
2. Let $f : A \to B$ and $g : B \to C$ be functions. Assume that $g \circ f$ is injective. Prove that f must be injective.
   - Assume $f(x_1) = f(x_2)$. Then $g(f(x_1) = g(f(x_2) = (g \circ f)x_1 = (g \circ f)x_2$ But $g \circ f$ is injective. Therefore $x_1 = x_2$.
3. Find a bijective map $f : \mathsf{R} \to (0,1)$ from all real numbers to the open interval $(-1,1)$. It is enough to sketch the graph of such a function.
   - A function like arctan does the job.
4. Define the divisibility relation | between natural numbers. Is there a natural number m such that for every $n \in \mathsf{N}$ one has that $n|m$?
   - $n|m$ iff $\exists_k(k \cdot n = m)$ We have $n \cdot 0 = m$ for every $n \in \mathsf{N}$. Thus $n|0$ for every $n \in \mathsf{N}$. (This assumes $0 \in N$, otherwise there is no such m)
5. Let $f : A \to B$ be any map. Define a relation $\sim_f$ on A by $\sim_f = \{(a_1,a_2)|f(a_1) = f(a_2)\}$. Explain why $\sim_f$ is an equivalence relation.
   - $f(x) = f(x),$ thus $\sim_f$ is reflexive. If $f(x) = f(y)$ then $f(y) = f(x),$ Thus $\sim_f$ is symmetric. If $f(x) = f(y), f(y) = f(z)$ then $f(x) = f(z).$ Thus $\sim_f$ is transitive.
6. Let $d : \mathsf{R}^2 \to \mathsf{R}, (x,y) \mapsto \sqrt{x^2 + y^2}$. What is the equivalence class $[(0,1)]$ with respect to $\sim_d$
   - $[(0,1)] = \{(x,y)|\sqrt{x^2+y^2} = 1\}$
7. Prove that f is injective iff $\sim_f$ is the equality on A.

- The function f is injective iff $f(x_1) = f(x_2) \iff x_1 = x_2 \iff x_1 \sim_f x_2$

8. Let E and F be two equivalence relations on the set A.

a. Prove that the intersection $E \cap F = \{(a,b)|aEb \text{ and } aFb\}$ is an equivalence relation.

- $(x,x) \in E$ as well $(x,x) \in F$ because E and F are reflexive. Thus $E \cap F$ is reflexive. Symmetry and Transitivity are similar.

b. Prove that in general the union $E \cup F = \{(a,b)|aEb \text{ or } aFb\}$ is not an equivalence. Here you have to give an example of two equivalence relations whose union fails to be transitive.

- We take two relations on $\mathbb{Z}$. Let E have the partition into even and odd integers. Let F be have the partition into non-negative integers and negative integers. Then $-4E2, 2F3$ but neither $(-4,3) \in E$ nor $(-4,3) \in F$. Because $(-4,2),(2,3) \in E \cup F$ we have that $E \cup F$ is not transitive. (A better choice is $A = \{1,2,3\}$ where E has the classes $\{1,2\},\{3\}$ and F the classes $\{1\}$ and $\{2,3\}$ then $1E2$ and $2F3$ but neither $1E3$ nor $1F3$.