April 18, 2016

Name:

Test 3, Key Math 3330

You have **80** minutes to complete the test. Each problem is worth **20** points. You cannot

use any books or notes.

1. Label each of the following statements as either true of false.
   a. The order of the identity element in any group is 1. **T**
   b. All subroups of a cyclic group are cyclic. **T**
   c. Two abelian groups of the same order are isomorphic. **F**
   d. If $G =< a >$ is of order $n > 2$ then $G =< a^{n-1} >$. **T**
   e. If H is a subgroup of the group G then only for $x = e$ the coset xH is a subgroup of G **T**.
   f. Let H be any subgroup of the group G. Then $xH = H$ if and only if $x \in H$. **T**
   g. If G is an abelian group then every subroup H of G is normal. **T**
   h. The subgroups $< e >$ and G are both normal subgroups of G. **T**
   i. Z and 2Z are isomorphic. **T**
   j. The group $\mathbb{Q}$ of rational numbers with addition is an infinite cyclic group. **F**
2. Prove that every cyclic group is a homomorphic image of the group $\mathbb{Z}$ of integers with addition.

   Proof. Let $G =< x >= \{x^n | n \in \mathbb{Z}\}$ The map $e : \mathbb{Z} \to G, n \mapsto x^n$ is homomorphic: $e(n+m) = x^{n+m} = x^n x^m = e(n)e(m), e(-n) = x^{-n} = (x^n)^{-1} = e(n)^{-1}, e(0) = x^0 = 1$.
3. What are k and l in $2\mathbb{Z} + 3\mathbb{Z} = k\mathbb{Z}, 2\mathbb{Z} \cap 3\mathbb{Z} = l\mathbb{Z}$?

   Answer: $2\mathbb{Z} + 3\mathbb{Z} = (2,3)\mathbb{Z} = \mathbb{Z}; 2\mathbb{Z} \cap 3\mathbb{Z} = [2,3]\mathbb{Z} = 6\mathbb{Z}$
4. Let $G =< a >$ be a cyclic group of order 8.
   a. Find the order of of each of the following a^2,a^3, a^4, a^5, a^6, a^7, a^8.

      Answer: $4, 8, 2, 8, 4, 8, 1$
   b. Find all distinct generators of $< a >$.

      Answer: a, a^3, a^7
5. Prove that a homomorphic image of an abelian group is abelian.

   Proof: Let $\varphi : G \twoheadrightarrow H$ be a surjective homomorphis from the abelian group G onto the group H. Let h_1, h_2 be any two elements of H. Then $h_1 = \varphi(g_1), h_2 = \varphi(g_2)$ and $h_1 h_2 = \varphi(g_1)\varphi(g_2) = \varphi(g_1 g_2) = \varphi(g_2 g_1) = \varphi(g_2)\varphi(g_1) = h_2 h_1$.
6. Let G be a group of order p^2 where p is a prime number. Prove that G contains a subgroup H of order p.

   Proof: Let $x \in G$ where $x \neq e$. Then by Lagrange's Theorem, the order of $< x >$ is either p or p^2. If $|< x >|= p^2$ then G is cyclic and must have a subgroup of order p. Actually, $< x^p >$ is of order p. If $|< x >|= p$ then we have found a subgroup of order p.