Problems and Comments on Relations

Chapter 8

Section 8.1, Problems: 1, 24, 30, 47, 53

Section 8.4, Problems: 1, 2, 3, 22, 23

Section 8.5, Problems: 3, 16, 41, 55, 57

Section 8.6, Problems: 5, 12, 15, 33, 62, 64

The idea of a relation between elements of a set A and a set B can be formalized by specifying a subset R of the Cartesian product $A \times B.$ Instead of saying that a and b are in the relation $R,$ or $aRb,$ we say that $(a,b) \in R \subseteq A \times B.$ On any set A we have the equality relation, $x = y,$ and this turns into the diagonal $\Delta = \{(a,a) | \, a \in A\}$. If $A = \mathbb{R},$ then Δ is the graph of the function $y = x.$ The order relation $x < y$ turns into the set $\{(x,y) | \, x < y\}$ and this is the set of points in the plane $\mathbb{R}^2$ which are below the line $y = x.$

A function $f : A \to B$ leads to a relation by saying that x and y are in the functional relation determined by $f,$ in case that $y = f(x).$ This relation is obviously represented by the graph of $f.$ Recall that $graph(f) = \{(x,y) | \, y = f(x)\} \subseteq A \times B.$

The main advantage of treating relations as sets is that we can apply all the common set theoretical operations, like union, intersection and complement. Of particular interest is also the inverse of a relation. If R is a relation between elements of A and $B,$ then the inverse relation R^{-1} is a relation between the elements of B and A :

$$R^{-1} = \{(b,a) | \, (a,b) \in R\}$$

For example, if $<$ is the order on $\mathbb{R}, (a,b) \in \; <$ iff $a < b$ then the inverse is the greater relation $>,$ because $a < b$ iff $b > a.$ You should pay special attention to functions. The inverse relation for a function is a function only if the function is invertible.

Of great importance is the composition of relations. As it is the case for functions, relations in order to be composable have to be connected. If $R \subseteq A \times B, S \subseteq B \times C$ then the composition of S and R is defined by: $S \circ R = \{(a,c) | \exists b \, (a,b) \in R, (b,c) \in S\}.$ In particular, we can form all powers R^n of R where R^0 is defined as $\Delta, R^1 = R, R^2 = R \circ R,$ and $R^{n+1} = R^n \circ R.$ Notice that the composition of composable relations is associative. Composition with a diagonal does not change a relation.

Notice that $(a,b) \in R^n$ iff there are

$$c_1, c_{2,}, \ldots, c_{n-1} \text{ such that } aRc_1, c_1Rc_2, \ldots, c_{n-1}Rb$$

$$R^* = \bigcup \{R^n | \, n \geq 1\}$$

is called the *connectivity relation* for $R.$ We have $(a,b) \in R^*$ iff there is a path in R from a to $b.$

If the relation is transitive, then one has $R^2 \subseteq R$: $(a,b) \in R$ and $(b,c) \in R,$ that is $(a,c) \in R^2$ yields $(a,c) \in R$. On the other hand, the meaning of $R^2 \subseteq R$ is that whenever one has some b such that $(a,b) \in R, (b,c) \in R$ one has that $(a,c) \in R$. But this is transitivity. *Hence, $R^2 \subseteq R$ iff R is transitive.*

If $R^2 \subseteq R$, then $R \circ R^2 = R^3 \subseteq R \circ R = R^2 \subseteq R,$ that is $R^3 \subseteq R$. That is, *R is transitive if and only if $R^n \subseteq R, n \geq 1$.* Thus R is transitive iff $R^* = R$. (Notice that $R \subseteq R^*$)

That the relation R is reflexive means $\Delta = R^0 \subseteq R$.

The relation R on A is symmetric if $R^{-1} = R$.

If R is any relation on $A,$ $R \subseteq A \times A,$ then adding the diagonal makes it reflexive. $R \cup \Delta$ is called the *reflexive closure* of A.

To make R symmetric, one needs to add R^{-1} to $R,$ $R \cup R^{-1}$ is called the *symmetric closure* of A.

To make R transitive, on has to add all powers R^n of R to R. The connectivity relation R^* is also called the *transitive closure* of R.

It is easy to see that the intersection of reflexive relations is reflexive. For any given relation R on A, there is is a relation which contains R and which is reflexive, namely $R \times R$. From this observation alone, one can conclude that there is a smallest relation that contains R, and which is reflexive. This relation then might be called the reflexive closure. We already know what this closure looks like.

Similarly, the intersection of symmetric relations is symmetric and $R \times R$ is symmetric. Hence, for any relation R there is a smallest symmetric relation that contains R. This relation is the symmetric closure of R.

The intersection of transitive relations is transitive and $R \times R$ is transitive. Hence for any relation R on A there must be a smallest transitive relation that contains $R,$ the transitive closure. It is R^*.

Pay attention to Lemma1, p. 501, in the book. If A is finite with n elements, then one needs only the calculation of the first n powers of R in order to calculate R^*.

Exercise *Show that the intersection of transitive relations is transitive and that*

$$\bigcap \{S \mid R \subseteq S, S \text{ transitive} \} = R^*$$

Equality is reflexive, we certainly have $a = a$, equality is symmetric, that is $a = b \Rightarrow b = a$ and equality is transitive, $a = b, b = c \Rightarrow a = c$. Equivalence relations are relations which share with equality these three properties. Congruence and similarity of triangles are equivalence relations between triangles. Triangles are congruent if the lengths of their sides are the same. Triangles are similar if the have the same angles. The idea of an equivalence between objects A and B (e.g., triangles A and B) is that A and Bhave certain characteristics (e.g., length of sides, angles) in common. If we define a function f on the set of triangles that assigns to a triangle A the lengths of its three sides, like $f(A) = (a = 3, b = 5, c = 5)$ then $A \equiv B$ iff $f(A) = f(B)$. The idea that objects are equivalent if a function defined for them agree can be vastly generalized:

Let $f : A \to B$ be any function. Define $\ker(f) = \{(a_1, a_2) \mid f(a_1) = f(a_2)\}$. Then $\ker(f)$ is an equivalence and called the *kernel* of the function $f.$.

Every equivalence relation E leads to a partition π_E. (a partition is a collection of non-empty subsets of $A,$ called classes, which are pairwise disjoint and whose union is A). Two elements are in the same class of π_E iff they are equivalent. And every partition π leads to an equivalence E_π where elements are equivalent if they belong to the same class. The relationship between equivalence relations and partitions is nearly tautological but requires a formal proof (Theorem 1, Theorem 2 on p. 510, 512).

For linear maps $T : U \to V$ between vector spaces, one identifies the class of the zero vector, $N = [0] = \{u|\ T(u) = 0\},$ with the kernel. One can do that because $N,$ also called the null space (for T), determines any other class: $[u] = u + N.$

If T is a temperature distribution function, then the equivalence classes for T are called isotherms.

If $N : C \to \{A,B,C,D,F\}$ is the function which assigns to a student in the class C his standing with respect to his grade, then C partitions into the classes of A-students, B-students etc.

Transitivity is a fundamental property of any concept of ordering. Anti-symmetry is equally important, one cannot have $a \le b$ and $b \le a$ unless $a = b.$ A relation $\le$ which is reflexive, transitive and anti-symmetric is called a *partial ordering*. If any two elements a and b are comparable, that is either $a \le b$ or $b \le a,$ then the partial order is called a *total order.*

For any subset S of a partially ordered set one can define that u is an *upper bound* for $S : u \ge s$ for all $s \in S.$ An upperbound for S which belongs to S is the *maximum* of S. If a set has a maximum then it is unique (Proof?). *Lower bounds* and *minima* are similarly defined.

If for a subset $S,$ the set of upper bounds is non-empty and has a minimum, then this least upper bound is called the *supremum* of $S.$ *Infimum* is similarly defined as the largest lower bound of a set $S.$ The open interval $(0,1)$of real numbers has infimum 0 and supremum 1.

It is an axiom for real numbers that any subset S of real numbers which has an upper bound, has a supremum. This is the *Least Upper Bound Axiom*.

A partially ordered set $(P,\le)$ is called *bounded* if it has a minimum as well as a maximum. A bounded partially ordered set is called a *complete lattice* in case that every non-empty subset S has a supremum and an infimum. For any set S the powerset $\mathbf{P}(S)$ with inclusion as partial ordering is an example of a complete lattice. The infimum of a set of subsets is its intersection, the supremum is its union.

A partially ordered set $(L,\le)$ is called a *lattice* if every finite subset has a supremum and an infimum. The natural numbers $(\mathrm{N},|)$, where | is the partial order given by divisibility, is a lattice. Here the infimum of $\{a,b\}$ is the $\gcd\{a,b\},$ the greatest common divisor. Recall that the greatest common divisor of a,b is the number d such that $d|a$ and $d|b.$ That is, d is a lower bound with respect to divisibility of a and $b,$ and if e is any other lower bound for a and $b,$ that is $e|a$ and $e|b,$ then $e|d.$ That is, d is the greatest lower bound for $\{a,b\}$. The lowest common multiple of $\{a,b\}$ is the supremum of $\{a,b\}$. The existence of and $\gcd$ is shown in any modern algebra course. It is not trivial and relies on the unique prime factorization theorem. The uniqueness part of this theorem is not trivial.

Any partial order on a finite set can be extended to a total order. This is called topological sorting and the book provides an algorithm for doing that. The following exercise will do the same and can be applied to infinite partial orderings.

Exercise *Let (A, R) be a partial ordering and assume that neither aRb nor bRa holds, i.e., $(a, b) \notin R, (b, a) \notin R$. Then the transitive closure of $R \cup \{(a, b)\}$ is antisymmetric*