MATH 4332

Homework 3
due Wednesday, February 6

Q1. Let $u(x) = \sum_{n=1}^{\infty} \frac{\cos 2\pi nx}{n^2}$. Show that (a) u is a continuous 1-periodic function on $\mathbb{R}$ ($u(x+n) = u(x)$, all $n \in \mathbb{Z}$), (b) if $v(x) = \frac{1}{2\pi} \sum_{n=1}^{\infty} \frac{\sin 2\pi nx}{n^3}$, then v is C^1 (continuously differentiable) and $v' = u$. Justify your arguments carefully, using appropriate results from the course.

Q2. Suppose that (f_n) is a sequence of continuous functions which is pointwise convergent to f on the open interval (a, b). Suppose that the convergence of (f_n) to f is *uniform* on every closed subinterval of (a, b). Does it follow that f is continuous on (a, b)? Proof or counterexample.

Q3. Consider the infinite series $\sum_{n=1}^{\infty} \frac{1}{1+n^2x}$.

(a) For what values of x is the series convergent?

(b) On what closed intervals does it converge uniformly.

(c) Is f continuous on the set of points where the series converges? (You might want to use the result of Q2).

Q4. Show that the series $\sum_{n=1}^{\infty} \frac{x}{n(1+nx^2)}$ is uniformly convergent on $\mathbb{R}$. (Hint: use the M-test: bound $|\frac{x}{n(1+nx^2)}|$.)

Q5. Let (u_n), (v_n) be sequences. For $p \geq 1$, let s_p denote the partial sum $\sum_{n=1}^{p} u_n$. Prove that for all $p \geq 1$

$$\sum_{n=1}^{p} u_n v_n = \sum_{n=1}^{p-1} s_n(v_n - v_{n+1}) + s_p v_p.$$

(Use induction.) Deduce that if (v_n) is a decreasing sequence of positive terms and there exist $c, C \in \mathbb{R}$ such that $c \leq s_p \leq C$ for all $p \geq 1$, then

$$cv_1 \leq \sum_{n=1}^{p} u_n v_n \leq Cv_1.$$

(Note: we will use this result later in the course. It is very useful.)