MATH 4332

Homework 6
due **Wednesday, March 12**

Q1. Define

$$
\begin{aligned}
f(x) &= x^2 \sin\left(\frac{\pi}{\sqrt{x}}\right), x > 0, \\
&= 0, \; x \leq 0.
\end{aligned}
$$

You may assume f is smooth on $x \neq 0$. Show that

1. f is continuous on $\mathbb{R}$.
2. f is differentiable at $x = 0$ and $f'(0) = 0$.
3. f' is continuous on $\mathbb{R}$.
4. f is not twice differentiable at $x = 0$.

What is the zero set $(f^{-1}(0))$ of f? Give two reasons why f is not analytic on $\mathbb{R}$.

Q2. Let $(I_n)_{n=1}^{\infty}$ be a sequence of non-empty, mutually disjoint open intervals and (α_n) be a sequence of strictly positive real numbers.

Suppose that (f_n) is a sequence of positive continuous functions on $\mathbb{R}$ such that for $n \geq 1$, (a) f_n is non-zero precisely on I_n and (b) the maximum value of f_n is α_n.

1. Is $\sum_{n=1}^{\infty} f_n$ always *pointwise* convergent on $\mathbb{R}$?
2. Find a necessary and sufficient condition on the sequence (α_n) that allows the M-test to be applied to $\sum_{n=1}^{\infty} f_n$.
3. Is it true that if the M-test does not apply, then $\sum_{n=1}^{\infty} f_n$ cannot be uniformly convergent on $\mathbb{R}$? If false, provide an example.
4. Denote the length of the interval I_n by ℓ_n. Show that if there exists $A > 0$ such that $\ell_n \geq A$ for all $n \geq 1$, then $\sum_{n=1}^{\infty} f_n$ is always continuous on $\mathbb{R}$.

Q3. Let $\psi : \mathbb{R} \to \mathbb{R}$ be the function which is defined to be zero for $x \leq 0$ and $\exp(-1/x)$ for $x > 0$. You may assume that (a) ψ is C^∞, (b) $\psi^{(n)}(0) = 0$, $n \geq 0$.

1. Find, using ψ, a C^∞ function e such that $e > 0$ on $(-\infty, 0) \cup (1, \infty)$ and $e \equiv 0$ on $[0, 1]$.
2. Find, using ψ, a C^∞ function f such that $f(0) = 0$, elsewhere $f < 0$ and $f^{(j)}(0) = 0$, $j \geq 0$.
3. Find, using ψ, a C^∞ function g such that the zero set of g is $\{\pm n^3 | n \in \mathbb{Z}\}$, elsewhere $g < 0$.
4. Find, using ψ, a C^∞ function h such that $h(x) = 0$, $x \leq 0$, and
   (a) $h(x) = n + 1$, if $x \in [2n + 1, 2n + 2]$, $n \geq 0$.
   (b) $h(x) \in (n, n + 1)$, if $x \in (2n, 2n + 1)$, $n \geq 0$.

   (You are advised to draw the graph first. One step at a time.)