

Uniform Convergence: Main results

We emphasize series.

THEOREM

Let $I \subset \mathbb{R}$ and suppose (u_n) is a sequence of continuous functions defined on I . If $u = \sum_{n=0}^{\infty} u_n$ is uniformly convergent on I , then $u \in C^0(I)$.

The proof is obtained by looking at the partial sums and applying the result for sequences.

THEOREM

Let $I \subset \mathbb{R}$ be an interval and suppose (u_n) is a sequence of continuous functions defined on I . If $u = \sum_{n=0}^{\infty} u_n$ is uniformly convergent on I and $a \in I$, then

$$\int_a^x u(t) dt = \sum_{n=0}^{\infty} \int_a^x u_n(t) dt, \quad (x \in I).$$

THEOREM

Let $I \subset \mathbb{R}$ be an open interval and suppose (u_n) is a sequence of C^1 functions defined on I . If $\sum_{n=0}^{\infty} u_n$ is pointwise convergent at one point of I and if $\sum_{n=0}^{\infty} u'_n$ is uniformly convergent on I , then $u = \sum_{n=0}^{\infty} u_n$ defines a C^1 -function on I and

$$u'(x) = \sum_{n=0}^{\infty} u'_n(x), \quad (x \in I).$$

M-test

Let $I \subset \mathbb{R}$ be an interval and suppose (u_n) is a sequence of continuous functions defined on I . If there exists a sequence (M_n) such that

- (1) $|u_n(x)| \leq M_n, x \in I, n \geq 0$.
- (2) $\sum_{n=0}^{\infty} M_n < \infty$.

then $\sum_{n=0}^{\infty} u_n$ is uniformly convergent on I .

Consider the power series $\sum_{n=0}^{\infty} a_n x^n$. Suppose that the series has radius of convergence R . (Recall that if the radius of convergence is R , then the series converges if $|x| < R$ and diverges if $|x| > R$.)

The key estimate is on the coefficients of a power series. If the radius of convergence is $R > 0$ and we choose $s \in (0, R)$, then there exists $M = M(s) \geq 0$ such that for all

$n \geq 0$,

$$|a_n| \leq \frac{M}{s^n}.$$

(Since $s < R$, $\sum a_n s^n < \infty$. A necessary condition for convergence of $\sum a_n s^n$ is that $a_n s^n \rightarrow 0$. It follows that $(|a_n s^n|)$ is bounded, say by M . Therefore $|a_n s^n| \leq M$, all $n \geq 0$.)

Consequences:

- (1) If $\sum_{n=0}^{\infty} a_n x^n$ has radius of convergence R , then for all $0 \leq r < R$, $\sum_{n=0}^{\infty} a_n x^n$ is uniformly convergent on $[-r, +r]$. It follows that $u(x) = \sum_{n=0}^{\infty} a_n x^n$ is continuous on $(-R, R)$.
- (2) If $\sum_{n=0}^{\infty} a_n x^n$ has radius of convergence R , then the radius of convergence of $\sum_{n=0}^{\infty} a_n \frac{x^{n+1}}{n+1}$ and of $\sum_{n=1}^{\infty} n a_n x^{n-1}$ are both equal to R .
- (3) A consequence of (1,2) is that if $u(x) = \sum_{n=0}^{\infty} a_n x^n$ has radius of convergence R , then u is a C^∞ function on $(-R, R)$ and the derivatives of u are obtained by term-by-term differentiation.

We indicate the proof of the second part of (2). Let $r \in (0, R)$ and set $s = \frac{R+r}{2} > r < R$. There exists $M = M(s)$ such that

$$|a_n| \leq \frac{M}{s^n}, n \geq 0.$$

Hence $|n a_n| \leq \frac{nM}{s^n}$, $n \geq 1$. Now for $x \in [-r, r]$ we have

$$|n a_n x^n| \leq \frac{nM}{s^n} r^n = nM \left(\frac{r}{s}\right)^n.$$

Since $\frac{r}{s} < 1$, $\sum_{n=1}^{\infty} nM \left(\frac{r}{s}\right)^n < \infty$ (for example, by the ratio test) and so, by the M-test, $\sum_{n=1}^{\infty} n a_n x^{n-1}$ is uniformly convergent on $[-r, r]$. Since this is true for all $0 < r < R$, $\sum_{n=1}^{\infty} n a_n x^{n-1}$ has radius of convergence at least R . If the radius of convergence is greater than R , we work backwards to deduce that the radius of convergence of $\sum_{n=0}^{\infty} a_n x^n$ is greater than R . Contradiction.