Riemannian Geometry
The Bochner- Weitzenbock formula

If we need to verify some tensor identity (or inequality) on Riemannina manifolds, we only need to choose, at every point, a suitable local coordinate, and verify the identity (or inequality) at the given point. In this handout, we will discuss how to make the choice of the local coordinate and prove (or re-prove) some useful formulas for the differential operators on the Riemannina manifolds M.

1 Normal (geodesic) coordinates

We first prove the following statement about the exitence of the *normal coordinates*: *For each point $p \in M$, there exists the* **normal coordinate** *$(U, x^1, \ldots, x^m)$ at p, i.e. we have $x^i(p) = 0$, $g_{ij}(p) = \delta_{ij}$ and $\Gamma_{ij}^k(p) = 0$ for $1 \leq i, j, k \leq m$.*

Proof: According the theorem about existence of the solution for (system) ODEs, for every point $P \in M$ and $\mathbf{v} \in T_P(M)$, there exists a unique geodesics $C(t, P, \mathbf{v})$ (or we just write $C_{\mathbf{v}}(t)$) such that $C_{\mathbf{v}}(0) = P$, $C'_{\mathbf{v}}(0) = \mathbf{v}$. Before we continue, we make the following very **important** remark: we always have $C_{\lambda\mathbf{v}}(t) = C_{\mathbf{v}}(\lambda t)$, becuase if we set $\boldsymbol{\alpha}(u) = C_{\mathbf{v}}(\lambda u)$, then

$$\frac{d\boldsymbol{\alpha}}{du} = \lambda \frac{dC_{\mathbf{v}}}{dt}, \quad \frac{d^2\boldsymbol{\alpha}}{du^2} = \lambda^2 \frac{d^2 C_{\mathbf{v}}}{dt^2}.$$

Thus, since $C_{\mathbf{v}}$ is a geodesic, $\boldsymbol{\alpha}$ is also a geodesic and stasfies that $\boldsymbol{\alpha}(0) = P, \boldsymbol{\alpha}'(0) = \lambda\mathbf{v}$, so by the uniquness, $C_{\mathbf{v}}(\lambda t) = C_{\lambda\mathbf{v}}(t)$. This proves the remark.

The *exponential mapping* $exp_P(\mathbf{v})$: $T_P(M) \to M$ is defined by $\mathbf{v} \mapsto C_{\mathbf{v}}(1)$. Then, we can prove that *there exists $\epsilon > 0$ such that exp_P: $B(O_P, \epsilon) \subset T_P(M) \to M$ is a diffeomorphism onto its image.* Let $U = exp_P(B(O_P, \epsilon))$. Take an orthonormal basis $\{\mathbf{e}_1, \ldots, \mathbf{e}_m\}$ for $T_P(M)$, this determines an associated coordinate $(x^1, \ldots, x^m)$. We verify that the coordinates $(U, x^1, \ldots, x^m)$ is a normal coordinate for P. First, we claim that

$$\frac{\partial}{\partial x^i}|_P = \mathbf{e}_i.$$

In fact, let $\phi = (x^1, \dots, x^m)$, and take any function f around P, we have (only veify for $i = 1$)

$$
\begin{aligned}
\frac{\partial f}{\partial x^1}|_P &= \frac{\partial f \circ \phi^{-1}}{\partial x^1}|_P \\
&= \frac{d}{dt} f \circ \phi^{-1}(t, 0, \dots, 0)|_{t=0} = \frac{d}{dt} f \circ C_{t\mathbf{e}_1}(1)|_{t=0} = \frac{d}{dt} f \circ C_{\mathbf{e}_1}(t)|_{t=0} = \mathbf{e}_1(f).
\end{aligned}
$$

So

$$\frac{\partial}{\partial x^i}|_P = \mathbf{e}_i.$$

This impiles that $g_{ij}(p) = \delta_{ij}$.

Next, we show that $\Gamma^k_{ij}(p) = 0$ for $1 \le i, j, k \le m$. Let $q = C_{\mathbf{v}}(1) \in U$ and let $C^i(t)$ be the coordinate components of $C_{\mathbf{v}}(t)$. Write $\phi(q) = (\xi^1, \dots, \xi^m)$. Then, since $C_{\mathbf{v}}(t) = C_{t\mathbf{v}}(1)$, we have that $C^i(t) = t\xi^i$. Since $C_{\mathbf{v}}(t)$ is geodesic, the geodesic equation implies that

$$\Gamma^k_{ij}(C_{\mathbf{v}}(t))\xi^i \xi^j = 0.$$

Let $t \longrightarrow 0$, we get

$$\Gamma^k_{ij}(C_{\mathbf{v}}(0))\xi^i \xi^j = 0.$$

Because ξ^i are arbitray, and $\Gamma^k_{ij} = \Gamma^k_{ji}$, this implies that $\Gamma^k_{ij}(p) = 0$. So the statement is proved.

2 Reproof of some formulas for Differential operators

In this notes, we always assume that M is a Riemannian manifold with the Levi-Civita connection $\bigtriangledown$, and $X, Y, ...$ are smooth vecor fields.

- **A basis formula for d.** Let (M, g) be a Riemannian manifold with the Levi-Civita connection $\bigtriangledown$. Let $\{e_i\}$ be a local frame field on M (i.e. a basis for $\Gamma(U, TM)$) and $\{\omega^i\}$ be its dual, i.e. $\omega^j(e_i) = \delta^j_i$. Then, for every smooth differential r-form θ,

$$d\theta = \sum_i \omega^i \wedge \bigtriangledown_{e_i} \theta. \tag{1}$$

Proof: First notice that it is independent of the choice of coordinates. So we choose normal coordinates x^i, i.e. we have $x^i(p) = 0$, $g_{ij}(p) = \delta_{ij}$ and $\Gamma^k_{ij}(p) = 0$ for $1 \leq i, j, k \leq m$. Let $\{e_i\}$ is a local **orthonormal** frame field on M with $e_i(p) = \frac{\partial}{\partial x_i}|_p$, and let $\{\omega^j\}$ be the dual to $\{e_i\}$. We claim that

$$(\bigtriangledown_{e_i}\omega^j)(p) = 0.$$

In fact, since $\delta_{jk} = \omega^j(e_k) = (e_k, \omega^j)$, we have

$$d\delta_{jk} = 0 = (\bigtriangledown e_k, \omega^j) + (e_k, \bigtriangledown\omega^j),$$

i.e. $\bigtriangledown\omega^j(e_k) = \omega^j(\bigtriangledown e_k)$, hence

$$\bigtriangledown_{e_i}\omega^j = -\sum_{k=1}^m \Gamma^j_{ik}\omega^k.$$

Thus we get, using $\Gamma^j_{ik}(p) = 0$,

$$(\bigtriangledown_{e_i}\omega^j)(p) = 0.$$

Therefore the claim holds.

Now, since $\bigtriangledown_{\partial/\partial x^i} dx^j = 0$, we have, for $\theta = f dx^{i_1} \wedge \cdots \wedge dx^{i_r}$,

$$\sum_i \omega^i \wedge \bigtriangledown_{e_i}\theta = \sum_i \frac{\partial f}{\partial x_i} dx^i \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_r} = d\theta.$$

- **The divergence operator.** A $(1,1)$-tensor T can be viewed as an endomorphism $T : V \to V$, and the trace of T is called the contraction of T, denoted by $trace(T)$ or $C^1_1(T)$. Let $X \in \Gamma(TM)$. Then $\bigtriangledown X$ is a smooth $(1,1)$-tensor field on M. Take a contraction of $\bigtriangledown X$, we get a smooth function on M. It is called the *divergenece of* X, i.e. $div(X) = C^1_1(\bigtriangledown X)$. The map $div : \Gamma(TM) \to C^\infty(M)$ given by $X \mapsto div(X)$ is called the *divergence operator*. In temrs of local coordinate $(U; x^i)$,

$$div(X) = \frac{1}{\sqrt{G}} \sum_{i=1}^m \frac{\partial}{\partial x^i}(\sqrt{G} X^i),$$

where $G = det(g_{ij}), g_{ij} = g(\partial/\partial x^i, \partial/\partial x^j), X^i = dx^i(X)$.

Solution: Choose normal coordinates x^i at $p \in M$ and write

$$X|_U = X^i \frac{\partial}{\partial x^i}.$$

Then, at point p,

$$\frac{1}{\sqrt{G}} \sum_{i=1}^{m} \frac{\partial}{\partial x^i}(\sqrt{G} X^i) = \sum_{i=1}^{m} \frac{\partial X^i}{\partial x^i},$$

where on the other hand, at point p,

$$div(X) = tr\{Y \to \bigtriangledown_Y X\} = \sum_{i=1}^{m} \bigtriangledown_{\frac{\partial}{\partial x^i}} X^i = \sum_{i=1}^{m} \frac{\partial X^i}{\partial x^i}.$$

So the formula holds.

- **The gradient of** f Let $f \in C^\infty(M)$, define a tangent vector field $grad(f)$ on M, by

$$g(grad(f), X) = df(X) = X(f),$$

for every smooth tangent vector field X. The tangent vector field $grad(f)$ is called the *gradient* of f. In local coordinate $(U; x^i)$,

$$grad(f) = \sum_{j=1}^{m} \left(\sum_{i=1}^{m} g^{ij} \frac{\partial f}{\partial x^i} \right) \frac{\partial}{\partial x^j},$$

where $G = det(g_{ij}), g_{ij} = g(\partial/\partial x^i, \partial/\partial x^j), (g^{ij}) = (g_{ij})^{-1}$.

- **Beltrami-Laplace operator** Let $f \in C^\infty(M)$, define $\triangle f = div(grad(f))$. It is called the *Laplace operator*. In local coordinate $(U; x^i)$,

$$\triangle f = \frac{1}{\sqrt{G}} \sum_{i=1}^{m} \frac{\partial}{\partial x^i} \left(\sum_{j=1}^{m} \sqrt{G} g^{ij} \frac{\partial f}{\partial x^j} \right),$$

where $G = det(g_{ij}), g_{ij} = g(\partial/\partial x^i, \partial/\partial x^j), (g^{ij}) = (g_{ij})^{-1}$.

- **Interior product** For any vector field X, $\iota(X)$ sends r-form to $r-1$ defined by, for every r-form ω and vector fields $Y_1, \ldots, Y_{r-1}$,

$$(\iota(X)\omega)(Y_1, \ldots, Y_{r-1}) = \omega(X, Y_1, \ldots, Y_{r-1}).$$

- Let η be its volume form of M. For every smooth tangent vector field X,
$$d(\iota(X)\eta) = div(X)\eta,$$
where $\iota(X)$ is the interior product.

  Proof: By definition, $\eta = \sqrt{G}dx^1 \wedge \cdots \wedge dx^m$, we claim $\iota(X)\eta = \omega$ where
$$\omega = \sum_{i=1}^{m}(-1)^{i+1}\sqrt{G}X^i dx^1 \wedge \cdots \wedge \hat{dx^i} \wedge \cdots \wedge dx^m.$$
We now prove it. Indep. of the choice of coordinates. Choose normal coordinates x^i. Then, at the point p, $\eta = dx^1 \wedge \cdots \wedge dx^m$ and $X = \sum_{j=1}^m X^j e_j$, hence
$$\iota(e_j)(\omega^1 \wedge \cdots \wedge \omega^m) = (-1)^{j+1}\omega^1 \wedge \cdots \hat{\omega^j} \wedge \cdots \wedge \omega^m.$$
This proves the claim. The rest of proof follows easily.

- **Divergence theorem**: *Let (M, g) be a compact oriented Riemannian manifold, then, for every smooth tangent vector field X,*
$$\int (divX)\eta = 0,$$
where η is the volume form.

- **The expression of the co-differential operator** δ: *Let $\{e_i\}$ be a local frame field on M compatible with the orientation of M. Let $g_{ij} = g(e_i, e_j)$, and $(g^{ij}) = (g_{ij})^{-1}$. Then the codifferential operator δ can be written as*
$$\delta\alpha = -\sum_{i,j=1}^{m} g^{ij}(\nabla_{e_i}\alpha), \qquad for\ every \quad \alpha \in \Lambda^r(M).$$

  If $\{e_i\}$ is **orthonormal**, *then we can write*
$$\delta = -\sum_{j=1}^{m}\iota(e_j)\nabla_{e_j}, \tag{2}$$

where $i(X)$ is the interior product operator, i.e. for every $\alpha \in \Lambda^r(M)$, and for every tangent vector fields $X_1, \ldots, X_{r-1}$.

$$\begin{aligned}
\delta\alpha(X_1, \ldots, X_{r-1}) &= -\sum_{j=1}^{m} \iota(e_j)(\bigtriangledown_{e_j}\alpha)(X_1, \ldots, X_{r-1}) \\
&= -\sum_{j=1}^{m} (\bigtriangledown_{e_j}\alpha)(e_j, X_1, \ldots, X_{r-1}).
\end{aligned}$$

Proof: For $p \in M$, choose the **normal coordinate** $(U, x^1, \ldots, x^m)$ at p. Let $\{e_i\}$ is a local **orthonormal** frame field on M with $e_i(p) = \frac{\partial}{\partial x_i}|_p$, and let $\{\omega^j\}$ be the dual to $\{e_i\}$. Then

$$(\bigtriangledown_{e_i}\omega^j)(p) = 0.$$

To prove

$$\delta(\alpha) = -\sum_{j=1}^{m} \iota(e_j)(\bigtriangledown_{e_j}\alpha). \tag{3}$$

We need only to verify it at each point $p \in M$. Since the operator is linear, without loss of generality, we assume that

$$\alpha = f\omega^1 \wedge \cdots \wedge \omega^r.$$

Hence

$$\begin{aligned}
\bigtriangledown_{e_j}\alpha &= \bigtriangledown_{e_j}(f)\omega^1 \wedge \cdots \wedge \omega^r + f \bigtriangledown_{e_j}(\omega^1 \wedge \cdots \wedge \omega^r) \\
&= e_j(f)\omega^1 \wedge \cdots \wedge \omega^r + f \bigtriangledown_{e_j}(\omega^1 \wedge \cdots \wedge \omega^r).
\end{aligned}$$

Using $(\bigtriangledown_{e_i}\omega^j)(p) = 0$, (only) **at the point** p, we have

$$\bigtriangledown_{e_j}\alpha = e_j(f)\omega^1 \wedge \cdots \wedge \omega^r.$$

Hence, **at the point** p, we have

$$\iota(e_j)(\bigtriangledown_{e_j}\alpha) = e_j(f)(\iota(e_j)(\omega^1 \wedge \cdots \wedge \omega^r)).$$

Because

$$\iota(e_j)(\omega^1 \wedge \cdots \wedge \omega^r) = (-1)^{j+1}\omega^1 \wedge \cdots \wedge \hat{\omega}^j \wedge \cdots \wedge \omega^r,$$

we have

$$\iota(e_j)(\nabla_{e_j}\alpha) = (-1)^{j+1}e_j(f)\omega^1 \wedge \cdots \wedge \hat{\omega}^j \wedge \cdots \wedge \omega^r.$$

This tells us, **at the point** p, that

$$-\sum_{j=1}^{m} \iota(e_j)(\nabla_{e_j}\alpha) = -\sum_j (-1)^{j+1}e_j(f)\omega^1 \wedge \cdots \wedge \hat{\omega}^j \wedge \cdots \wedge \omega^r. \quad (4)$$

We now calculate the left-hand side. By defintion, $\delta = (-1)^{n(r+1)+1} * d *$. We have,

$$\begin{aligned}\delta(\alpha) &= (-1)^{n(r+1)+1} * d * (\alpha) = (-1)^{n(r+1)+1} * d * (f\omega^1 \wedge \cdots \wedge \omega^r) \\ &= (-1)^{n(r+1)+1} * d(f\omega^{r+1} \wedge \cdots \wedge \omega^m) \\ &= (-1)^{n(r+1)+1} * (\sum_j e_j(f)\omega^j \wedge \omega^{r+1} \wedge \cdots \wedge \omega^m)\end{aligned}$$

Note that

$$*(\omega^j \wedge \omega^{r+1} \wedge \cdots \wedge \omega^m) = (-1)^{(r-1)(n-r-1)+(r-j)}\omega^1 \wedge \cdots \wedge \hat{\omega}^j \wedge \cdots \wedge \omega^r,$$

hence

$$\begin{aligned}\delta(\alpha) &= \sum_j (-1)^{n(r+1)+1}(-1)^{(r-1)(n-r-1)+(r-j)}e_j(f)\omega^1 \wedge \cdots \wedge \hat{\omega}^j \wedge \cdots \wedge \omega^r \\ &= \sum_j (-1)^j e_j(f)\omega^1 \wedge \cdots \wedge \hat{\omega}^j \wedge \cdots \wedge \omega^r.\end{aligned}$$

Comparing the above identity with (4), we conclude that (2) holds **at every point** p. Hence the theroem holds.

- **The operator δ can be viewed as a generalization of the divergenece**: In fact, let X be a vector field, then $\delta(\alpha_X) = -div(X)$, where α_X is the 1-form defined by $\alpha_X(Y) = g(X, Y)$ for every smooth tangent vector field Y.

  Proof: Again, they are indep. of the choice of coordinates. Choose normal coordinates x^i. Then, at point p. Let $\{e_i\}$ is a local **orthonormal** frame field on M with $e_i(p) = \frac{\partial}{\partial x_i}|_p$, and let $\{\omega^j\}$ be the dual to $\{e_i\}$.

Let $X = \sum X^i e_j$, then $\alpha_X = \sum X^j \omega^j$, $div(X) = -\sum e_i(X^j)$. On the other hand,

$$\delta(\alpha_X) = -\sum \bigtriangledown_{e_i}(\alpha_X)(e_i).$$

At point p,

$$\bigtriangledown_{e_i}(\alpha_X) = \sum e_i(X^j)\omega^j + \sum X^j \bigtriangledown_{e_i} \omega^i = \sum e_i(X^j)\omega^j.$$

Hence $\bigtriangledown_{e_i}(\alpha_X)(e_i) = e_i(X^i)$. This proves the theorem.

3 Bochner-Weitzenbock Formulas

The Bochner-Weitzenbock Formulas, sometimes refered to as the Bochner technique, is one of the most important technique in the theory of geometric analysis.

We want to express the Hodge-Laplace operator $\triangle$ in terms of the Levi-Civita connection $\bigtriangledown$.

We first consider the function case. For functions, i.e. 0-form f, by defintion, $g(grad(f), Y) = Y(f)$ and $df(Y) = Y(f)$, so $\alpha_{grad(f)} = df$. Hence, from above, $\delta(df) = -div(grad(f))$. Therefore,

$$\triangle f = \delta d(f) = -div(grad(f)) = -tr \bigtriangledown^2 f,$$

where, $tr\bigtriangledown^2 := \sum_{i=1}^m \bigtriangledown_{e_i}\bigtriangledown_{e_i}$ for an orthonormal basis $\{e_i\}$.

In general, let $\{e_i\}$ be a local frame for a Riemannian manifold (M, g), define

$$tr\bigtriangledown^2 : \Lambda^r(M) \to \Lambda^r(M)$$

as

$$tr \bigtriangledown^2 (\alpha) = g^{ij}(\bigtriangledown_{e_i} \bigtriangledown_{e_j} - \bigtriangledown_{\bigtriangledown_{e_i} e_j})\alpha,$$

for every $\alpha \in \Lambda^r(M)$.

For 1-form α, we have

$$\triangle\alpha(X) = -tr \bigtriangledown^2 \alpha(X) + r(\alpha^\#, X),$$

where r is the Ricci tensor of (M, g), and $\alpha^{\#}$ is the vector field defined by $g(\alpha^{\#}, Y) = \alpha(Y)$.

Proof. The right hand side is independent of the choice of our orthonormal frame field. Therefore, we only need to verify it at every point $p \in M$. To do so, we choose normal coordinates centered at p and put at p,

$$e_i = \frac{\partial}{\partial x^i}.$$

Let ω^j be its dual frame. Then, always at p,

$$\bigtriangledown_{e_i} e_j = 0.$$

This also gives, at p,

$$\bigtriangledown_{e_i} \omega^j = 0.$$

Using (1) and (2), we then have at p,

$$\begin{aligned}
(\delta d\alpha)(X) &= (\delta(\omega^j \wedge \bigtriangledown_{e_j} \alpha))(X) \\
&= -(\bigtriangledown_{e_i} (\omega^j \wedge \bigtriangledown_{e_j} \alpha))(e_i, X) \\
&= -(\omega^j \wedge \bigtriangledown_{e_i} \bigtriangledown_{e_j} \alpha)(e_i, X) \\
&= -\bigtriangledown_{e_i} \bigtriangledown_{e_i} \alpha(X) + X^j \bigtriangledown_{e_i} \bigtriangledown_{e_j} \alpha(e_i).
\end{aligned}$$

$$\begin{aligned}
(d\delta\alpha)(X) &= (\omega^j \bigtriangledown_{e_j} (\delta\alpha))(X) \\
&= X^j \bigtriangledown_{e_j} (\delta\alpha) \\
&= -X^j \bigtriangledown_{e_j} \bigtriangledown_{e_i} \alpha(e_i).
\end{aligned}$$

Hence,

$$\triangle\alpha(X) = -tr\bigtriangledown^2\alpha(X) + X^j \bigtriangledown_{e_i} \bigtriangledown_{e_j} \alpha(e_i) - X^j \bigtriangledown_{e_j} \bigtriangledown_{e_i} \alpha)(e_i) = X^j R(e_i, e_j)\alpha(e_i),$$

where where $X = X^k e_k$ and

$$R(e_i, e_j)\alpha = (\bigtriangledown_{e_i} \bigtriangledown_{e_j} - \bigtriangledown_{e_j} \bigtriangledown_{e_i})\alpha.$$

We now claim that

$$X^j R(e_i, e_j)\alpha(e_i) = r(\alpha^{\#}, X).$$

In fact, write $\alpha = \alpha_k \omega^k$, then $\alpha^\# = \alpha_k e_k$, and

$$\begin{aligned} X^j R(e_i, e_j)\alpha(e_i) &= X^j \alpha_k R(e_i, e_j)\omega^k(e_i) \\ &= -X^j \alpha_k R(e_j, e_i)\omega^k(e_i) \\ &= -X^j \alpha_k R_{kmji}\omega^m(e_i) \\ &= -X^j \alpha_k R_{kiji} \\ &= -r(\alpha^\#, X). \end{aligned}$$

This proves the statement.

We now derive the formula for a general r-form. For the purpose, we define the second covariant derivative as

$$\bigtriangledown^2_{XY} = \bigtriangledown_X \bigtriangledown_Y - \bigtriangledown_{\bigtriangledown_X Y}\,.$$

Weitzenbock's formula. *Let $e_1, \ldots, e_m$ be a local* **orthonormal** *frame field, with the dual frame field $\omega^1, \ldots, \omega^m$. Then the Hodge-Laplace operator $\triangle$ acting on r-differential forms is given by*

$$\triangle = -\sum_{i=1}^{m} \bigtriangledown^2_{e_i e_i} - \sum_{i,j} \omega^i \wedge \iota(e_j) R(e_i, e_j).$$

Proof. The right hand side is independent of the choice of our orthonormal frame field. Therefore, we only need to verify it at every point $p \in M$. To do so, we choose normal coordinates centered at p and put at p,

$$e_i = \frac{\partial}{\partial x^i}.$$

Then, always at p,

$$\bigtriangledown_{e_i} e_j = 0.$$

Hence,

$$\bigtriangledown^2_{e_i e_i} = \bigtriangledown_{e_i} \bigtriangledown_{e_i},$$

and also

$$[\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}] = 0.$$

Therefore

$$R(e_i, e_j) = \bigtriangledown_{e_i} \bigtriangledown_{e_j} - \bigtriangledown_{e_j} \bigtriangledown_{e_i}.$$

Using (1) and (2), we then have at p,

$$\begin{aligned}
\delta d &= -\iota(e_j) \bigtriangledown_{e_j} (\omega^i \wedge \bigtriangledown_{e_i}) \\
&= -\iota(e_j)(\omega^i \wedge \bigtriangledown_{e_j} \bigtriangledown_{e_i}) \\
&= -\bigtriangledown_{e_j} \bigtriangledown_{e_j} + \omega^i \wedge \iota(e_j) \bigtriangledown_{e_j} \bigtriangledown_{e_i}.
\end{aligned}$$

To calcluale $d\delta$, we note that, since $\bigtriangledown_{e_i} \omega^j = 0$,

$$\iota(e_j) \bigtriangledown_{e_i} = \bigtriangledown_{e_i} \iota(e_j).$$

Hence,

$$\begin{aligned}
d\delta &= -\omega^i \wedge \bigtriangledown_{e_i}(\iota(e_j) \bigtriangledown_{e_j}) \\
&= -\omega^i \wedge \iota(e_j) \bigtriangledown_{e_i} \bigtriangledown_{e_j}.
\end{aligned}$$

So

$$\triangle = d\delta + \delta d = -\sum_{i=1}^{m} \bigtriangledown^2_{e_i e_i} - \sum_{i,j} \omega^i \wedge \iota(e_j) R(e_i, e_j).$$

This proves the statement.

Remak: On functions, i.e. 0-form f, we have

$$R(e_i, e_j) f = f R(e_i, e_j) 1 = 0.$$

Hence, we have, for a local orthonormal frame field,

$$\triangle f = -\sum_{i=1}^{m} \bigtriangledown^2_{e_i e_i} f.$$

Theorem. *For any smooth differential form η (r-forms),*

$$-\triangle |\eta|^2 = 2|\bigtriangledown \eta|^2 + 2 < \sum_i \bigtriangledown^2_{e_i e_i} \eta, \eta >,$$

where

$$|\bigtriangledown \eta|^2 = \sum_i |\bigtriangledown_{e_i} \eta|^2.$$

Remark: Consider the Eucliean case $\mathbf{R}^2$ and $\eta = f$. Then

$$- \triangle |f|^2 = 2f_x f_x + 2f_y f_y + 2f_{xx} f + 2f_{yy} f = 2|\bigtriangledown f|^2 + 2 < f_{xx} + f_{yy}, f > .$$

The theorem is motivated by it, and can be derived by direct computation. We only verify it for 1−form η. We only need to verify it at every point $p \in M$. Let $\{e_1, \ldots, e_m\}$ be a local orthonormal frame fields and $\{\omega_1, \ldots, \omega_m\}$ be the cofarme fields. Write

$$\eta = \sum_i \eta_i \omega^i,$$

then, at point p, using the formula of $\triangle$ for smooth functions,

$$-\frac{1}{2} \triangle |\eta|^2 = \sum_i < \bigtriangledown_{e_i} \bigtriangledown_{e_i} \eta, \eta > + \sum_i |\bigtriangledown_{e_i} \eta|^2.$$

This finishes the proof.

Combining the both, we get

Bochner's formula. *Let $e_1, \ldots, e_m$ be a local* **orthonormal** *frame field, with the dual frame field $\omega^1, \cdots, \omega^m$. Let η be a r-form on M. Then*

$$\frac{1}{2} \triangle |\eta|^2 =< \triangle\eta, \eta > -|\bigtriangledown \eta|^2 + < \omega^i \wedge \iota(e_j) R(e_i, e_j)\eta, \eta > .$$

Proof: Choose normal coordinate. From above,

$$\frac{1}{2} \triangle |\eta|^2 = -|\bigtriangledown \eta|^2 - \sum_i < \bigtriangledown^2_{e_i e_i} \eta, \eta > .$$

And from the Weitzenbock's formula,

$$\triangle\eta = -\sum_{i=1}^m \bigtriangledown^2_{e_i e_i} \eta - \sum_{i,j} \omega^i \wedge \iota(e_j) R(e_i, e_j)(\eta).$$

Hcece,

$$-\sum_{i=1}^m \bigtriangledown^2_{e_i e_i} \eta = \triangle(\eta) + \sum_{i,j} \omega^i \wedge \iota(e_j) R(e_i, e_j)(\eta).$$

Sbustituting it into above, we get

$$\frac{1}{2} \triangle |\eta|^2 =< \triangle\eta, \eta > -| \bigtriangledown \eta|^2 + < \omega^i \wedge \iota(e_j) R(e_i, e_j)\eta, \eta > .$$

For any smooth one-form, we get

Corollary *Let* $e_1, \ldots, e_m$ *be a local* **orthonormal** *frame field, with the dual frame field* $\omega^1, \cdots, \omega^m$*. Let* η *be a smooth one-form, then*

$$\frac{1}{2} \triangle |\eta|^2 =< \triangle\eta, \eta > -| \bigtriangledown \eta|^2 - r(\eta, \eta),$$

where $| \bigtriangledown \eta|^2 := \sum_i < \bigtriangledown_{e_i}\eta, \bigtriangledown_{e_i}\eta >$*, and writing* $\eta = \sum f_i\omega^i$*,*

$$r(\eta, \eta) := \sum_{i,j} r(f_ie_i, f_je_j) = \sum_{i,j} f_if_jr(e_i, e_j).$$

Proof: We only need to compute, for 1-form η,

$$\begin{aligned} < \eta, \omega^i \wedge \iota(e_j)R(e_i, e_j)\eta > &= < f_l\omega^l, \omega^i \wedge \iota(e_j)R(e_i, e_j)f_k\omega^k > \\ &= -f_lf_k < \omega^l, \omega^i \wedge \iota(e_j)R(e_i, e_j)\omega^k > \\ &= -f_lf_k < \omega^l, \omega^i \wedge \iota(e_j)R_{kmij}\omega^m > \\ &= -f_lf_k < \omega^l, R_{kjij}\omega^i > \\ &= -f_lf_kR_{kjlj} \\ &= -f_lf_kR_{kl} \\ &= -r(\eta, \eta). \end{aligned}$$

Theorem(Bochner) *Let* M *be a compact Riemannian manifold. If* M *has positive Ricci curvautre, then* M *has no nontrivial harmonic 1-form, thus,*

$$H^1_{dR}(M, \mathbf{R}) = \{0\}.$$

Proof. We integrate the formula above, and using the divergence theorem,

$$0 = -\int_M \triangle|\omega|\eta = 2\int_M(| \bigtriangledown \omega|^2 + r(\omega, \omega))\eta.$$

By the assuption, the integrand on the right hand side is pointwise non-negative. It therefore has to vanish identically. Hence $r(\omega, \omega) = 0$, which implies that $\omega = 0$ since the Ricci curvautre on M is positive. This proves the statement.

4 Proof of Garding's inequality

Theorem Garding's inequality: *There exist constant $c_1, c_2 > 0$, such that for every $f \in \bigwedge^*(M)$, we have*

$$(\triangle f, f) \geq c_1 \|f\|_1^2 - c_2 \|f\|_0^2.$$

Proof. For every $f \in \bigwedge^*(M)$, from Bochner's formula,

$$\begin{aligned} < \triangle f, f > &= \frac{1}{2} \triangle |f|^2 + |\bigtriangledown f|^2 - < \omega^i \wedge \iota(e_j) R(e_i, e_j) f, f > \\ &\geq \frac{1}{2} \triangle |f|^2 + |\bigtriangledown f|^2 - a_1 |f|^2, \end{aligned}$$

where a_1 is a constant independent of f. Note that the last inequality holds because $< \omega^i \wedge \iota(e_j) R(e_i, e_j) f, f >$ does not depend on the derivative(differential) of f (Note: although $R(e_i, e_j)$ depends on the derivative(differential), but since $R(e_i, e_j)(\alpha f) = \alpha R(e_i, e_j) f$ for every $\alpha \in C^{\inf}(M)$, so $< \omega^i \wedge \iota(e_j) R(e_i, e_j) f, f >$ is a quadratic form on $\bigwedge^r(M)$, its corfficients only depend on M, and since M is compact, such constant a_1 exists). Taking the integration on M and by definition, we get

$$(\triangle f, f) \geq c_1 \|f\|_1^2 - c_2 \|f\|_0^2 + \frac{1}{2} \int_M \triangle |f|^2 \eta.$$

But by Stokes theorem,

$$\int_M \triangle |f|^2 \eta = 0.$$

This proves the Garding's inequality.