§5.3 **Surface Theory with Differential Forms**

1 Differential forms on $\mathbf{R}^n$, Click here to see more details

Differential forms provide an approach to multivariable calculus (Click here to see more detailes) that is independent of coordinates.

Let U be an open set in $\mathbf{R}^n$. A **differential 0-form** ("zero form") is defined to be a smooth function f (here smooth means that f is differentiable at any order) on U.

If $\mathbf{v} \in \mathbf{R}^n$, then f has a directional derivative $D_{\mathbf{v}} f$ (see "introduction to differentiable function" in Chapter one), which is another function on U whose value at a point $p \in U$ is the rate of change (at p) of f in the $\mathbf{v}$ direction:

$$D_{\mathbf{v}} f(p) = \frac{\mathrm{d}}{\mathrm{d}t} f(p + tv)\Big|_{t=0}.$$

In particular, if $\mathbf{v} = \mathbf{e}_j$, where $\{\mathbf{e}_1 = (1, 0, \ldots, 0), \mathbf{e}_2 = (0, 1, \ldots, 0), ..., \mathbf{e}_n = (0, \ldots, 1)\}$ is the standard basis of $\mathbf{R}^n$, then

$$D_{\mathbf{e}_j} f = \frac{\partial f}{\partial x_j},$$

is the partial derivative of f with respect to the jth coordinate function, where $x_1, x_2, ...x_n$ are the coordinate functions on U. By their very definition, partial derivatives depend upon the choice of coordinates: if new coordinates $y_1, y_2, ...y_n$ are introduced, then

$$\frac{\partial f}{\partial x^j} = \sum_{i=1}^{n} \frac{\partial y^i}{\partial x^j} \frac{\partial f}{\partial y^i}.$$

The first idea leading to differential forms is the **observation that** $D_{\mathbf{v}} f(p)$ **is a linear function of** $\mathbf{v}$:

$$D_{\mathbf{v}+\mathbf{w}} f(p) = D_{\mathbf{v}} f(p) + D_{\mathbf{w}} f(p), \quad D_{c\mathbf{v}} f(p) = c D_{\mathbf{v}} f(p)$$

for any vectors $\mathbf{v}, \mathbf{w}$ and any real number c. This linear map from $\mathbf{R}^n$ to $\mathbf{R}$ is denoted df_p and called the **(exterior) derivative** of f at p. Thus $df_p(\mathbf{v}) = D_{\mathbf{v}} f(p)$. The object df can be viewed as a function on U, **whose value at** p **is not a real number, but the linear map** df_p, in other words, df assigns, at every point $p \in U$, a linear map df_p from $\mathbf{R}^n$ to $\mathbf{R}$. This is just a special case of (general) differential 1-forms. Since any vector $\mathbf{v}$ is a linear combination $\sum v_j \mathbf{e}_j$ of its components, df is uniquely determined by $df_p(\mathbf{e}_j)$ for each j and each $p \in U$ (i.e. $df_p(\mathbf{v}) = \sum v_j df_p(\mathbf{e}_j)$) which are just the partial derivatives of f on U. Thus df provides **a way of encoding the partial derivatives of** f. It can be decoded by noticing that the coordinates $x_1, x_2, ...x_n$ are themselves functions on U (x_j maps each point in U the j-th coordinate), and so define differential 1-forms $dx_1, dx_2, ...dx_n$ (most time, we also write it as $dx^1, dx^2 2, ...dx^n$), in other words,

$$dx^j(\mathbf{e}_i) = \delta_{ij}, \quad or dx^j(\mathbf{v}) = v_j, \quad \text{for } \mathbf{v} = (v_1, \ldots, v_n) \in \mathbf{R}^n, \tag{5.1.1}$$

In general, for any 0-form f,

$$df = \sum_{i=1}^{n} \frac{\partial f}{\partial x^i} \mathrm{d}x^i. \tag{5.1.2}$$

The meaning of this expression is given by evaluating both sides at an arbitrary point p: on the right hand side, the sum is defined "pointwise", so that

$$\mathrm{d}f_p = \sum_{i=1}^{n} \frac{\partial f}{\partial x^i}(p) \mathrm{d}x^i|_p.$$

Applying both sides to $\mathbf{e}_j$, the result on each side is the j-th partial derivative of f at p (note: $\mathrm{d}x^i|_p$ is actually independent of p, i.e. $\mathrm{d}x^i|_p = \mathrm{d}x^i$).

Example. Let $f = e^{x^2+y}$, then

$$df = \frac{\partial f}{\partial x}\mathrm{d}x + \frac{\partial f}{\partial y}\mathrm{d}y = 2xe^{x^2+y}dx + e^{x^2+y}dy.$$

We now give the general definition of $1-$forms on $U \subset \mathbf{R}^n$:

Definition 5.1.1 *A $1-$form ϕ on $U \subset \mathbf{R}^n$ (either $n = 2$ or $n = 3$) assigns, for every $p \in U \subset \mathbf{R}^n$, a is a linear map $\phi|_p : \mathbf{R}^n \to \mathbf{R}$.*

Remarks:

(1) In linear algebra, given a vector space V, the set of all linear maps $f : V \to \mathbf{R}$ is called the dual space of V. So the alternative definition of $1-$form ϕ is that *the $1-$form ϕ on $U \subset \mathbf{R}^n$ (either $n = 2$ or $n = 3$) assigns, for every $p \in U \subset \mathbf{R}^n$, an element $\phi_p \in \mathbf{R}^{n*}$, where $\mathbf{R}^{n*}$ is the dual space of $\mathbf{R}^n$.*

(2) In particular, the $1-$forms $dx^1, \ldots, dx^n$ are defined by the property that for (see (5.1.1)) any vector $\mathbf{v} = (v_1, \ldots, v_n) \in \mathbf{R}^n$,

$$dx^i|_p(\mathbf{v}) = v_i.$$

The dx^i form a basis for the space of all $1-$forms on $\mathbf{R}^n$, so **any** $1-$form ϕ on $U \subset \mathbf{R}^n$ may be expressed in the form

$$\phi = \sum_{i=1}^{n} \phi_i dx^i, \tag{5.1.3}$$

where $\phi_1, \ldots, \phi_n$ are functions on U. If $\mathbf{v} = (v_1, \ldots, v_n) \in \mathbf{R}^n$, then

$$\phi_p(\mathbf{v}) = \sum_{i=1}^{n} f_i(p) v_i.$$

(3) Note that $\{dx^1, \ldots, dx^n\}$ is in fact the standard basis of the dual space $\mathbf{R}^{n*}$, it is dual to the standard basis $\{\mathbf{e}_1 = (1, 0, \ldots, 0), \mathbf{e}_2 = (0, 1, \ldots, 0), ..., \mathbf{e}_n = (0, \ldots, 1)\}$ of $\mathbf{R}^n$.

Example. $\phi = \sin x dx + y^2 dy$ is a 1-form on $\mathbf{R}^2$.

You might ask the following question: *given a differential 1-form ϕ on U, when does there exist a function f on U such that $\phi = df$?* By comparing (5.1.2) and (5.1.3) reduces this question to the search for a function f whose partial derivatives $\partial f / \partial x_i$ are equal to n given functions ϕ_i. For $n > 1$, such a function does not always exist: any smooth function f satisfies

$$\frac{\partial^2 f}{\partial x^i \partial x^j} = \frac{\partial^2 f}{\partial x^j \partial x^i},$$

so it will be impossible to find such an f unless

$$\frac{\partial \phi_j}{\partial x^i} - \frac{\partial \phi_i}{\partial x^j} = 0.$$

for all i and j.

Differential $2-$ **forms on** $\mathbf{R}^n$.

For two $1-$forms ϕ, ψ, define the *wedge product* $\phi \wedge \psi$ as follows, for $p \in \mathbf{R}^n$, $\phi|_p \wedge \psi|_p : \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}$ is given by, for every $(\mathbf{v}, \mathbf{w}) \in \mathbf{R}^n \times \mathbf{R}^n$,

$$\phi|_p \wedge \psi|_p(\mathbf{v}, \mathbf{w}) = \phi(\mathbf{v})\psi(\mathbf{w}) - \psi(\mathbf{v})\phi(\mathbf{w}). \tag{5.1.4}$$

The wedge product is a way to produce 2−forms from the given two 1− forms.

By the definition, we have $\phi \wedge \psi = -\psi \wedge \phi$, and $\phi \wedge \phi = 0$.
A basis for the 2-forms on $\mathbf{R}^n$ is given by the set

$$\{dx^{i_1} \wedge dx^{i_2} : 1 \leq i_1 < i_2 \leq n\}.$$

Any 2-forms Ω on U can be expressed in the form

$$\Omega = \sum_{i_1 < i_2} f_{i_1, i_2} dx^{i_1} \wedge dx^{i_2},$$

where f_{i_1,i_2} are functions on U. In an alternative definition,

Definition 5.1.2 *A 2-form Ω on $U \subset \mathbf{R}^n$ assigns, at ach point $p \in U$, an alternating bilinear mapping $\Omega|_p : \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}$. Here, "alternating" means that $\Omega|_p(\mathbf{v}, \mathbf{w}) = -\Omega|_p(\mathbf{w}, \mathbf{v})$, and "bilinear" means that $\Omega|_p$ is linear in both $\mathbf{v}$ and $\mathbf{w}$. We can also definition the concept of the k-form in a similar way as*

$$\sum_{i_1 < i_2 < \cdots < i_k} f_{i_1, i_2, \cdots, i_k} dx^{i_1} \wedge \cdots \wedge dx^{i_k}.$$

The *exterior derivative* takes that takes k−forms to $(k+1)$−forms. It is defined in a similar way as above. For example, if $\phi = \sum_{|I|=k} f_I dx^{i_1} \wedge \cdots dx^{i_k}$, then the **exterior derivative** $d\phi$ of ϕ is the $(k+1)$− form which is given by

$$d\phi = \sum_{|I|=k} df_I \wedge dx^{i_1} \wedge \cdots dx^{i_k}. \tag{5.1.5}$$

If ϕ is a p−form and ψ is a q−form, then the Leibniz rule takes the form

$$d(\phi \wedge \psi) = d\phi \wedge \psi + (-1)^p \phi \wedge d\psi.$$

Example. Let $\omega = f dx + g dy + h dx$ be a 1-form on $U \subset \mathbf{R}^3$, then

$$d\omega = df \wedge dx + dg \wedge dy + dh \wedge dz = (g_x - f_y) dx \wedge dy + (h_y - g_z) dy \wedge dz + (f_z - h_x) dz \wedge dx.$$

Very Important Theorem: $d^2 = 0$.

Proof. We only check for 0-forms, i.e. for functions. In fact, by definition,

$$d(df) = d\left(\sum_{j=1}^{n} \frac{\partial f}{\partial x^i} dx^i\right) = \sum_{i,j} \frac{\partial^2 f}{\partial x^i \partial x^j} dx^j \wedge dx^i$$

$$= \sum_{i<j} \left(\frac{\partial^2 f}{\partial x^j \partial x^i} - \frac{\partial^2 f}{\partial x^i \partial x^j} \right) dx^j \wedge dx^i = 0.$$

This verifies the functions case. The general result follows from the similar method.

pull-backs: Given a map $F = (x_1, x_2, x_3) : U \subset \mathbf{R}^2 \to \mathbf{R}^3$, and $\omega = f dx + g dy + h dx$ be a 1-form on $U \subset \mathbf{R}^3$, then we can define $F^*\omega$, the pull-back of ω by F as

$$F^*\omega = f \circ F dx_1 + g \circ F dx_2 + h \circ F dx_3$$

$$= f \circ F \left(\frac{\partial x_1}{du} du + \frac{\partial x_1}{dv} dv \right) + g \circ F \left(\frac{\partial x_2}{du} du + \frac{\partial x_2}{dv} dv \right) + h \circ F \left(\frac{\partial x_3}{du} du + \frac{\partial x_3}{dv} dv \right),$$

where (u, v) is the coordinate system on $\mathbf{R}^2$. Note that $F^*\omega$ is a differential form on $U \subset \mathbf{R}^2$, and it is easy to check that (do it by yourself)

$$F^* d\omega = dF^*\omega.$$

2 Differential forms on surfaces

We now define the concept of the differential forms on the surface M in $\mathbf{R}^3$. To do so, we need to look at the tangent space $T_p(M)$. Recall that **when** M **is flat**, i.e. $M = U$ or $M = \mathbf{R}^3$, then its tangent space T_pM is the whole space $\mathbf{R}^2$, i.e. $T_pU \cong \mathbf{R}^2$ for every point $p \in U$. So when we look at the definition of 1-forms ϕ on $U \subset \mathbf{R}^3$, actually we read it as follows: a 1-form ϕ on $U \subset \mathbf{R}^3$ assgins, at each poitn $p \in U$, a linear map $\omega|_p : T_p(U) \cong \mathbf{R}^2 \to \mathbf{R}$, i.e., $\phi|_p \in T_p^*(U) \cong \mathbf{R}^{2*}$.

We now give the general definition of differential forms on M.

Definition 5.2.2 *Given a surface M in $\mathbf{R}^3$, a (differential) $1-$form ω on M assigns, at every $p \in M$, a linear map $\omega|_p : T_p(M) \to \mathbf{R}$, i.e., $\omega|_p \in T_p^*(M)$.*

A (differential) $2-$form Ω on M assigns, at every point $p \in M$, a two form $\Omega|_p \in \bigwedge^2 T_p^*(M)$.

Every $k-$form on M is always zero for $k \geq 3$.

For a smooth function f on M, the **exterior** derivative of f is the $1-$form df with the property that for any $p \in M$, $\mathbf{v}_p \in T_p(M)$,

(5.2.1) $$df_p(\mathbf{v}_p) = D_{\mathbf{v}}(f)(p),$$

where $D_{\mathbf{v}}(f)(p)$ is the directional derivative of f at p with respect to the direction $\mathbf{v}_p$ (see (3.4.1) for the definition of the directional derivative). So as we indicated before, df provides a way of encoding **all** the directional derivative of f at p.

Let $\boldsymbol{\sigma} : U \subset \mathbf{R}^2 \to \mathbf{R}^3$ be a parametrization of M. Then the coordinates u, v on $\mathbf{R}^2$ be also regarded as functions on M sending the point $\boldsymbol{\sigma}(u, v)$ to u and v respectively. Hence, du and dv are well-defined 1-forms on M (more precisely on $\boldsymbol{\sigma}(U)$). It can be easily checked by definition that

(5.2.2) $$du|_p(\sigma_u|_p) = 1, du|_p(\sigma_v|_p) = 0, \quad dv|_p(\sigma_u|_p) = 0, dv|_p(\sigma_v|_p) = 1.$$

Hence $\{du, dv\}$ is dual to $\{\boldsymbol{\sigma}_u, \boldsymbol{\sigma}_v\}$.

For any smooth function f on M, In terms of parametrization $\boldsymbol{\sigma} : U \subset \mathbf{R}^2 \to \mathbf{R}^3$ of M, we can write

(5.2.3). $$df = D_{\boldsymbol{\sigma}_u} f du + D_{\boldsymbol{\sigma}_v} f dv$$

We we write $f(u, v) = f \circ \boldsymbol{\sigma}(u, v)$, since $D_{\boldsymbol{\sigma}_u} f(p) = (\partial f \partial u)(u_0, v_0)$, $D_{\boldsymbol{\sigma}_v} f(p) = (\partial f \partial u)(u_0, v_0)$, where $\boldsymbol{\sigma}(u_0, v_0) = p$, we have

(5.2.4) $$df = f_u du + f_v dv$$

where $f_u = \partial f \partial u, f_v = \partial f \partial v$. More precisely, we should write (5.2.4) as $\boldsymbol{\sigma}^* df = f_u du + f_v dv$, but we simply write it as in (5.2.4) without the danger of confusion. In other words, if we work everything on $\mathbf{R}^2$ by the pulling back through $\boldsymbol{\sigma}$, then we can regard du, dv as the standard 1-forms on $\mathbf{R}^2$ which is dual to $\{(1, 0), (0, 1)\}$.

In terms of the local parametrization $\boldsymbol{\sigma} : U \subset \mathbf{R}^2 \to \mathbf{R}^3$ of M, any $1-$form ω on M can be written as

$$\omega = adu + bdv$$

where a, b are functions on $\boldsymbol{\sigma}(U)$. Similarly, every $2-$form Ω can be locally (on $\boldsymbol{\sigma}(U)$) written as

$$\Omega = Adu \wedge dv,$$

where A is a function on $\boldsymbol{\sigma}(U)$.

For every 1-form ω on M, the **exterior** derivative of ω is a 2-form, which is defined in a similar way as in the previous section, by $\omega = adu + bdv$, then $d\omega = da \wedge du + db \wedge dv$. (you can check that it is independent of the choice of the choices of paramertizations). It is easy to check that $d^2 = 0$. The exterior operator d has the following important property: $d^2 = 0$, i.e. for every function f on M, $d(d(f)) = 0$.

3 The method of moving frames for curves

Let $\mathbf{x}(t) = (x_1(t), x_2(t), x_3(t))$ be a space curve. Let $\mathbf{e}_1$ be the unit-tangent vector. Let $\mathbf{e}_2$ such that $de_1/ds = \kappa e_2$. $\mathbf{e}_2$ is called the *principal normal* and κ is the curvature. Let $\mathbf{e}_3 = \mathbf{e}_1 \times \mathbf{e}_2$ is the *binormal*. Then $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ form an orthonormal basis(Frenet frame). Write

$$d\mathbf{e}_i = \sum_{j=1}^{3} \omega_{ij}\mathbf{e}_j,$$

where ω_{ij} are 1-forms. Since $< \mathbf{e}_i, \mathbf{e}_j >= \delta_{ij}$, form $d < \mathbf{e}_i, \mathbf{e}_j >= 0$, we have $\omega_{ij} = -\omega_{ji}$. Hence the matrix $(\omega)_{ij}$ is a 3×3 skew-symmetric matrix, whose entries are differential 1-forms. From the skew-symmetric, we have $\omega_{jj} = 0$. From the selection of $\mathbf{e}_1, \mathbf{e}_2$ and $\mathbf{e}_3$, we have $\omega_{13} = \omega_{31} = 0$. Hence, we have (Frenet formula):

$$\frac{d\mathbf{e}_1}{ds} = \kappa\mathbf{e}_2$$

$$\frac{d\mathbf{e}_2}{ds} = -\kappa\mathbf{e}_1 + \tau\mathbf{e}_3$$

$$\frac{d\mathbf{e}_3}{ds} = -\tau\mathbf{e}_2.$$

4 The method of moving frames for surfaces

1. Stucture equations

I suggest you to read the appendix below before you start this section.

Let M be a surface and $\boldsymbol{\sigma} : U \to \mathbf{R}^3$ be a local parametrization of M. Recall that the vectors $\{\boldsymbol{\sigma}_u|_p, \boldsymbol{\sigma}_v|_p\}$ is a basis for $T_p(M)$. Let $\mathbf{e}_1(p), \mathbf{e}_2(p)$ be an orthonormal basis for $T_p(M), p \in U$ (such orthonormal basis always exists by applying Gram-Schmidt orthonormalization procedure) and let $\mathbf{e}_3(p) = \mathbf{n}(p)$ be the unit normal(Gauss map). The **key point** of this section is that we are working on the **orthonormal basis**, **NOT** just the basis $\{\boldsymbol{\sigma}_u|_p, \boldsymbol{\sigma}_v|_p\}$. The basis $\{\mathbf{e}_1(p), \mathbf{e}_2(p), \mathbf{e}_3(p)\}$ serves as a moving frame for $\mathbf{R}^3$, where $\mathbf{e}_1(p), \mathbf{e}_2(p)$ is an orthonormal basis for $T_p(M)$.

Let $\boldsymbol{\sigma} : U \to \mathbf{R}^3$ be a local parametrization of M. Consider $d\boldsymbol{\sigma} = \boldsymbol{\sigma}_u du + \boldsymbol{\sigma}_v dv$. Since $\{\mathbf{e}_1(p), \mathbf{e}_2(p)\}$ is a basis for $T_p(M)$, we write

(5.3.1) $$\boldsymbol{\sigma}_u = \lambda_1\mathbf{e}_1 + \lambda_2\mathbf{e}_2, \quad \sigma_v = \lambda_3\mathbf{e}_1 + \lambda_4\mathbf{e}_4.$$

Hence we can rewrite $d\boldsymbol{\sigma}$ as

(5.3.2) $$d\boldsymbol{\sigma} = \omega_1\mathbf{e}_1 + \omega_2\mathbf{e}_2,$$

where ω_1, ω_2 are differential $1-$forms on M, $\omega_1 = \lambda_1 du + \lambda_3 dv$, $\omega_1 = \lambda_2 du + \lambda_4 dv$. We claim that $\{\omega_1, \omega_2\}$ is dual to $\{\mathbf{e}_1, \mathbf{e}_2\}$. To prove the claim, we first note that, form (5.3.1) $d\boldsymbol{\sigma}(\boldsymbol{\sigma}_u) = \boldsymbol{\sigma}_u$, and $d\boldsymbol{\sigma}(\boldsymbol{\sigma}_v) = \boldsymbol{\sigma}_v$, so $d\boldsymbol{\sigma}(\mathbf{v}) = \mathbf{v}$ for every tangent vector. Then the claim can be derived by using (5.3.2) and $d\boldsymbol{\sigma}(\mathbf{e}_1) = \mathbf{e}_1$ and $d\boldsymbol{\sigma}(\mathbf{e}_2) = \mathbf{e}_2$. The differential $1-$forms ω_1, ω_2 keep track of how our point moves around on M.

Next we want to see how the frame itself twists, we will define 1-forms $\omega_{ij}, i, j = 1, 2, 3$. Consider $d\mathbf{e}_i$, the exterior derivative of $\mathbf{e}_i$. Note that $d\mathbf{e}_i$ is a vector-valued (the image is in $\mathbf{R}^3$) differential $1-$form, and since $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ is linearly independent, so it is a basis for $\mathbf{R}^3$. Therefore we can

(5.3.4) $$d\mathbf{e}_i = \sum_{j=1}^{3} \omega_{ij}\mathbf{e}_j,$$

where $\omega_{ij}, i, j = 1, 2, 3$ are differential 1-forms. There are total nine of such 1-forms. First we claim that $\omega_{ij} = -\omega_{ji}$. In fact, since $\mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij}$, by differentiating, we have, $d\mathbf{e}_i \cdot \mathbf{e}_j + \mathbf{e}_i \cdot d\mathbf{e}_j = 0$. This implies that $\omega_{ij} = -\omega_{ji}$. Hence, we have only three "meaningful" 1-forms ω_{13}, ω_{23} and ω_{12}. Others are just zero. If $\mathbf{v}_p \in T_p(M)$, then $\omega_{ij}|_p(\mathbf{v}_p)$ tells us how fast $\mathbf{e}_i$ is twisting towards $\mathbf{e}_j$ at p as we move with velocity $\mathbf{v}_p$. Together with the above two 1-forms above, we obtain, in total, **five** 1-forms: $\omega_1, \omega_2, \omega_{13}, \omega_{23}, \omega_{12}$.

To summarize, we have the following **Equations for moving frame**

(5.3.5) $$d\boldsymbol{\sigma} = \omega_1\mathbf{e}_1 + \omega_2\mathbf{e}_2,$$

(5.3.6) $$d\mathbf{e}_1 = \omega_{12}\mathbf{e}_2 + \omega_{13}\mathbf{e}_3,$$

(5.3.7) $$d\mathbf{e}_2 = \omega_{21}\mathbf{e}_1 + \omega_{23}\mathbf{e}_3,$$

(5.3.8) $$d\mathbf{e}_3 = \omega_{31}\mathbf{e}_1 + \omega_{32}\mathbf{e}_2,$$

where $\omega_{ij} = -\omega_{ji}$.

Recall that the shape operator (see section 3.3) is $S_p = -d\mathbf{n} = -d\mathbf{e}_3$, so the shape operator is embodied in the equation

$$d\mathbf{e}_3 = \omega_{31}\mathbf{e}_1 + \omega_{32}\mathbf{e}_2 = -(\omega_{13}\mathbf{e}_1 + \omega_{23}\mathbf{e}_2).$$

We now calculate the matrix of the shape operator S_p respect to the basis $\{\mathbf{e}_1, \mathbf{e}_2\}$. In fact,

$$S_p(\mathbf{e}_1) = -d\mathbf{e}_3(\mathbf{e}_1) = -\omega_{31}(\mathbf{e}_1)\mathbf{e}_1 - \omega_{32}(\mathbf{e}_1)\mathbf{e}_2 = \omega_{13}(\mathbf{e}_1)\mathbf{e}_1 + \omega_{23}(\mathbf{e}_1)\mathbf{e}_2,$$

$$S_p(\mathbf{e}_2) = -d\mathbf{e}_3(\mathbf{e}_2) = \omega_{13}(\mathbf{e}_2)\mathbf{e}_1 + \omega_{23}(\mathbf{e}_2)\mathbf{e}_2.$$

Thus, by the definition, the matrix of the shape operator S_p respect to the basis $\{\mathbf{e}_1, \mathbf{e}_2\}$ is

$$\begin{pmatrix} \omega_{13}(\mathbf{e}_1) & \omega_{13}(\mathbf{e}_2) \\ \omega_{23}(\mathbf{e}_1) & \omega_{23}(\mathbf{e}_2) \end{pmatrix}.$$

Recall that the **the Gauss curvature** is the determinant of the matrix of S_p (see chapter 4), so we get

$$K = det(S_p) = \omega_{13}(\mathbf{e}_1)\omega_{23}(\mathbf{e}_2) - \omega_{13}(\mathbf{e}_2)\omega_{23}(\mathbf{e}_1) = (\omega_{13} \wedge \omega_{23})(\mathbf{e}_1, \mathbf{e}_2).$$

Since $\omega_{13} \wedge \omega_{23}$ is a 2-form, and the vector space of the two forms has dimension one (as we noted sarlier) we can write

$$\omega_{13} \wedge \omega_{23} = \lambda \omega_1 \wedge \omega_2.$$

So from above,

$$K = (\omega_{13} \wedge \omega_{23})(\mathbf{e}_1, \mathbf{e}_2) = \lambda \omega_1 \wedge \omega_2(\mathbf{e}_1, \mathbf{e}_2) = \lambda.$$

So we have a **very important** expression for the Gauss curvature

(5.3.9). $$\omega_{13} \wedge \omega_{23} = K\omega_1 \wedge \omega_2.$$

Most of our results will come from the following:

Theorem 5.2.1(**Structure Equations**):

(5.3.10) $$d\omega_1 = \omega_{12} \wedge \omega_2; \quad d\omega_2 = \omega_1 \wedge \omega_{12};$$

and

(5.3.11) $$d\omega_{ij} = \sum_{k=1}^{3} \omega_{ik} \wedge \omega_{kj}.$$

Proof: Use the property that $d^2 = 0$ and using (5.3.5), we have

$$0 = d(d\boldsymbol{\sigma}) = d(\omega_1 \mathbf{e}_1) + d(\omega_2 \mathbf{e}_2).$$

This derives the first two equations by using (5.3.6) and (5.3.7). Also, using (5.3.6), (5.2.7), (5.3.8) and $d^2 = 0$, we get

$$d(d\mathbf{e}_i) = d\omega_{ij} - \omega_{ij} \wedge d\mathbf{e}_j = 0.$$

This derives the second equation.

The structure equations give:

Gauss Equation: $d\omega_{12} = -\omega_{13} \wedge \omega_{23}$,

Mainardi-Codazzi Equation: $d\omega_{13} = \omega_{12} \wedge \omega_{23}$; $d\omega_{23} = -\omega_{12} \wedge \omega_{13}$.

From (5.3.9), we have

$$\omega_{13} \wedge \omega_{23} = K\omega_1 \wedge \omega_2.$$

Hence we get

$$d\omega_{12} = -K\omega_1 \wedge \omega_2.$$

This provides a **new** and simple proof of Gauss's theorem egregium (see section 5.1), since from above, we see that K only depends on ω_1 and ω_2, so the Gauss curvature is an *intrinsic* quantity!

Example Under the $\boldsymbol{\sigma} : U \to \mathbf{R}^3$ be a local parametrization of M with $E = G = \frac{4}{1+u^2+v^2}$, $F = 0$ (for example, on the $\mathbf{R}^2$ with the metric $g = \frac{4}{(1+x^+y^2)^2}(dx^2 + dy^2)$). Calculate the Gauss curvature.

Solution. Write $A = 1 + u^2 + v^2$. Then $\mathbf{e}_1 = \frac{A}{2}\boldsymbol{\sigma}_u$, $\mathbf{e}_2 = \frac{A}{2}\boldsymbol{\sigma}_v$. Since $\{\omega_1, \omega_2\}$ is dual to $\{\mathbf{e}_1, \mathbf{e}_2\}$, we have

$$\omega_1 = \frac{2}{A}du, \omega_2 = \frac{2}{A}dv.$$

To calculate the Gauss curvature, we need to find out the connection form ω_{12}. We use the structure equations $d\omega_1 = omega_{12} \wedge \omega_2, d\omega_2 = \omega_1 \wedge omega_{12}$ to find out ω_{12}. From $\omega_1 = \frac{2}{A}du$, we have

$$d\omega_1 = d\left(\frac{2}{A}\right) \wedge du = 2\frac{-dA}{A^2} \wedge du = \frac{4v}{A^2}du \wedge dv.$$

Writing $omega_{12} = adu + bdv$, and noting that $\omega_2 = \frac{2v}{A}dv$, from $d\omega_1 = omega_{12} \wedge \omega_2$ we find out that $a = 2y/A$. Similarly, we can get $b = -2u/A$. Hence

$$\omega_{12} = \frac{2v}{A}du - \frac{2v}{A}dv = v\omega_1 - u\omega_2.$$

Now we use $d\omega_{12} = K\omega_1 \wedge \omega_2$ to find out K. Since

$$d\omega_{12} = dv \wedge \omega_1 + vd\omega_1 - du \wedge \omega_2 - ud\omega_2$$

$$= \frac{A}{2}\frac{2}{A}dv \wedge \omega_1 + \frac{4v^2}{A^2}du \wedge dv - \frac{A}{2}\frac{2}{A}du \wedge \omega_2 + \frac{4u^2}{A^2}du \wedge dv$$

$$= -\frac{A}{2}\omega_1 \wedge \omega_2 + (u^2 + v^2)\omega_1 \wedge \omega_2 = -\omega_1 \wedge \omega_2.$$

Hence $K \equiv -1$.

2. Further look of Gauss equation and the Codazzi equations.

To see why the equation $d\omega_{12} = -\omega_{13} \wedge \omega_{23}$ is the same as the Gauss equation we derived before (in section 5.1), we consider a orthogonal parametrization, i.e. $F = 0$. In this case, $\mathbf{e}_1 = \mathbf{e}_u/\sqrt{E}, \mathbf{e}_2 = \mathbf{e}_v/\sqrt{G}$. Then, $\omega_1 = \sqrt{E}du, \omega_2 = \sqrt{G}dv$ (since $\{\omega_1, \omega_2\}$ is the dual basis to $\mathbf{e}_1, \mathbf{e}_2$). We now calculate ω_{12}. Write $\omega_{12} = Adu + bdv$, we need to determine A and B. From the structure equation,

$$-\omega_2 \wedge \omega_{12} = d\omega_1 = -(\sqrt{E})_v du \wedge dv.$$

Also

$$-\omega_2 \wedge \omega_{12} = -(\sqrt{G}dv) \wedge (Adu + Bdv) = \sqrt{G}Adu \wedge dv.$$

Hence

$$\sqrt{G}A = -(\sqrt{E})_v.$$

This implies that

$$A = -\frac{(\sqrt{E})_v}{\sqrt{G}}.$$

Similarily, we get

$$B = \frac{(\sqrt{G})_u}{\sqrt{E}}.$$

Hence

$$\omega_{12} = -\omega_{21} = -\frac{(\sqrt{E})_v}{\sqrt{G}}du + \frac{(\sqrt{G})_u}{\sqrt{E}}dv.$$

$$d\omega_{12} = \left[\left(\frac{(\sqrt{E})_v}{\sqrt{G}}\right)_v + \left(\frac{(\sqrt{G})_u}{\sqrt{E}}\right)_u\right] du \wedge dv.$$

On the other hand, from a direct computation, we can get

$$\omega_{13} = (d\mathbf{e}_1) \cdot \mathbf{e}_3 = \frac{e}{\sqrt{E}}du + \frac{f}{\sqrt{E}}dv,$$

$$\omega_{23} = (d\mathbf{e}_2) \cdot \mathbf{e}_3 = \frac{f}{\sqrt{G}} du + \frac{g}{\sqrt{G}} dv.$$

So

$$\omega_{13} \wedge \omega_{32} = -\omega_{13} \wedge \omega_{23} = -\frac{eg - f^2}{\sqrt{EG}} du \wedge dv.$$

Hence, Gauss equation $d\omega_{12} = -\omega_{13} \wedge \omega_{2,3}$ is equivalent to

$$-\left[\left(\frac{(\sqrt{E})_v}{\sqrt{G}} \right)_v + \left(\frac{(\sqrt{G})_u}{\sqrt{E}} \right)_u \right] = \frac{eg - f^2}{\sqrt{EG}},$$

which is the same as the Gauss equation we derived before in the case $F = 0$ (see section 5.1).

Similarily, we can verify the Codazzi equations listed above are the same as what we have derived earlier.

3. Normal curvature and geodesic curvature revisited.

Let $\boldsymbol{\sigma} : U \to M$ be a parametrization, and let C be a curve on M given by $\boldsymbol{\alpha}(s) = \boldsymbol{\sigma}(u(s), v(s))$ where s is the arc-lenght parameter. Let $\mathbf{T}(s) = \boldsymbol{\alpha}'(s)$ be the tangent vector to C, and let $\bar{\mathbf{e}}_1(s) = \mathbf{T}(s), \bar{\mathbf{e}}_2(s) = \mathbf{n}(s) \times \mathbf{T}(s), \bar{\mathbf{e}}_3(s) = \mathbf{n}(s)$, where $\mathbf{n}$ is the unit normal to the surface M. Then, $\{\bar{\mathbf{e}}_1(s), \bar{\mathbf{e}}_2(s), \bar{\mathbf{e}}_3(s)\}$ is an orthnormal moving frame along C (which is called *Darboux frame*). Then we have

$$\frac{d\boldsymbol{\alpha}(s)}{ds} = \bar{\mathbf{e}}_1(s),$$

$$\frac{d\bar{\mathbf{e}}_1(s)}{ds} = \kappa_g \bar{\mathbf{e}}_2(s) + \kappa_n \bar{\mathbf{e}}_3(s),$$

$$\frac{d\bar{\mathbf{e}}_2(s)}{ds} = -\kappa_g \bar{\mathbf{e}}_1(s) + \tau_g(s) \bar{\mathbf{e}}_3(s),$$

$$\frac{d\bar{\mathbf{e}}_3(s)}{ds} = -\kappa_n \bar{\mathbf{e}}_1(s) - \tau_g(s) \bar{\mathbf{e}}_2(s),$$

where κ_g is the geodesic curvature, κ_n is the normal curvature and $\tau_g(s)$ is called the geodesic torsion.

Take an orthonormal moving frame (Darboux frame) $\{\bar{\mathbf{e}}_1, \bar{\mathbf{e}}_2, \bar{\mathbf{e}}_3\}$ on M with $\bar{\mathbf{e}}_3 = \mathbf{n}$, such that the restriction of this frame to the curve C is $\{\bar{\mathbf{e}}_1(s), \bar{\mathbf{e}}_2(s), \bar{\mathbf{e}}_3(s)\}$ We write

$$d\bar{\mathbf{e}}_1 = \bar{\omega}_{12} \bar{\mathbf{e}}_2 + \bar{\omega}_{13} \bar{\mathbf{e}}_3,$$

$$d\bar{\mathbf{e}}_2 = \bar{\omega}_{21}\bar{\mathbf{e}}_1 + \bar{\omega}_{23}\bar{\mathbf{e}}_3,$$

$$d\bar{\mathbf{e}}_3 = \bar{\omega}_{31}\bar{\mathbf{e}}_1 + \bar{\omega}_{32}\bar{\mathbf{e}}_2,$$

where $\bar{\omega}_{ij} = -\bar{\omega}_{ji}$. We have the following theorem:

Theorem 5.2.2 *Let C be a curve on M, then*

$$\kappa_g = \bar{\omega}_{12}(\bar{\mathbf{e}}_1), \kappa_n = \bar{\omega}_{13}(\bar{\mathbf{e}}_1), \tau_g = \bar{\omega}_{23}(\bar{\mathbf{e}}_1).$$

Proof: Since $\boldsymbol{\alpha}(s) = \boldsymbol{\sigma}(u(s), v(s))$,

$$\bar{\mathbf{e}}_1 = \mathbf{T} = \frac{d\boldsymbol{\alpha}(s)}{ds} = \frac{du}{ds}\cdot\boldsymbol{\sigma}_u + \frac{dv}{ds}\cdot\boldsymbol{\sigma}_v.$$

Hence, for $1 \le i, j \le 3$,

$$\begin{aligned}
\bar{\omega}_{ij}(\mathbf{T}) &= < d\bar{\mathbf{e}}_i(\mathbf{T}), \bar{\mathbf{e}}_j > \\
&= < \frac{du}{ds} d\bar{\mathbf{e}}_i(\boldsymbol{\sigma}_u) + \frac{dv}{ds} d\bar{\mathbf{e}}_i(\boldsymbol{\sigma}_v), \bar{\mathbf{e}}_j > \\
&= < \frac{du}{ds}\bar{\mathbf{e}}_{i,u} + \frac{dv}{ds}\bar{\mathbf{e}}_{i,v}, \bar{\mathbf{e}}_j > \\
&= < \frac{d\bar{\mathbf{e}}_i}{ds}, \bar{\mathbf{e}}_j > .
\end{aligned}$$

Hence

$$\kappa_g = \bar{\omega}_{12}(\mathbf{T}), \kappa_n = \bar{\omega}_{13}(\mathbf{T}), \tau_g = \bar{\omega}_{23}(\mathbf{T}).$$

This finishes the proof.

We now re-derive the formula of κ_g in terms of the orthogonal paramerization. Let $\mathbf{x} : U \to S$ be a orthogonal parametrization, i.e $F = 0$ (where $\{E, F, G\}$ is its first fundamental form). Let C be a curve on S and let $\{\bar{\mathbf{e}}_1, \bar{\mathbf{e}}_2, \bar{\mathbf{e}}_3\}$ be the Darboux frame. Then, form above

$$\kappa_g = \bar{\omega}_{12}(\bar{\mathbf{e}}_1).$$

On the other hand, consider another (natural) orthonormal frame: i.e. let $\mathbf{e}_1 = \mathbf{x}_u/\sqrt{E}$, $\mathbf{e}_2 = \mathbf{x}_v/\sqrt{G}$, $\mathbf{e}_3 = \mathbf{e}_3 = \mathbf{n}$. Then the frame $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ is also an orthonormal moving frame. Let ω_1, ω_2 be the dual of $\mathbf{e}_1, \mathbf{e}_2$ and let ω_{12} be the connection form. Then, as we derived before,

$$\omega_{12} = -\omega_{21} = -\frac{(\sqrt{E})_v}{\sqrt{G}}du + \frac{(\sqrt{G})_u}{\sqrt{E}}dv.$$

Next, we want to derive a relationship between $\bar{\omega}_{12}$ and ω_{12}. To do so, let θ be the angle from $\mathbf{x}_u$ to $\bar{\mathbf{e}}_1 = \mathbf{T}$, then we have

$$\begin{pmatrix} \bar{\mathbf{e}}_1 \\ \bar{\mathbf{e}}_2 \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \end{pmatrix}.$$

Since $\{\omega_1, \omega_2\}$ is the dual basis of $\{\mathbf{e}_1, \mathbf{e}_2\}$(resp., $\{\bar{\omega}_1, \bar{\omega}_2\}$ is the dual basis of $\{\bar{\mathbf{e}}_1, \bar{\mathbf{e}}_2\}$), hence

$$\begin{aligned} \bar{\omega}_1 &= \cos\theta\omega_1 + \sin\theta\omega_2 \\ \bar{\omega}_2 &= -\sin\theta\omega_1 + \cos\theta\omega_2 \\ \bar{\omega}_{12} &= \omega_{12} + d\theta. \end{aligned}$$

Hence

$$\bar{\omega}_{12} = \omega_{12} + d\theta = -\frac{(\sqrt{E})_v}{\sqrt{G}}du + \frac{(\sqrt{G})_u}{\sqrt{E}}dv + d\theta.$$

Since

$$\mathbf{e}_1 = \mathbf{T} = \frac{d\boldsymbol{\alpha}(s)}{ds} = \frac{du}{ds}\mathbf{x}_u + \frac{dv}{ds}\mathbf{x}_v,$$

we have

$$\kappa_g = \bar{\omega}_{12}(\bar{\mathbf{e}}_1) = -\frac{(\sqrt{E})_v}{\sqrt{G}}du(\mathbf{T}) + \frac{(\sqrt{G})_u}{\sqrt{E}}dv(\mathbf{T}) + d\theta(\mathbf{T}) = -\frac{(\sqrt{E})_v}{\sqrt{G}}\frac{du(s)}{ds} + \frac{(\sqrt{G})_u}{\sqrt{E}}\frac{dv(s)}{ds} + d\theta(\mathbf{T}).$$

Now $d\theta = \theta_u du + \theta_v dv$, so

$$d\theta(\mathbf{T}) = \theta_u \frac{du(s)}{ds} + \theta_v \frac{dv(s)}{ds} = \frac{d\theta}{ds}.$$

Hence

$$\kappa_g = -\frac{(\sqrt{E})_v}{\sqrt{G}}\frac{du(s)}{ds} + \frac{(\sqrt{G})_u}{\sqrt{E}}\frac{dv(s)}{ds} + \frac{d\theta}{ds}.$$

Therefore we re-proved the following Liouville theorem:

Theorem 5.2.3 *Let $\mathbf{x} : U \to S$ be a orthogonal parametrization, i.e $F = 0$ (where $\{E, F, G\}$ is its first fundamental form). Let C be a curve on S. Then*

$$\kappa_g = -\frac{(\sqrt{E})_v}{\sqrt{G}}\frac{du(s)}{ds} + \frac{(\sqrt{G})_u}{\sqrt{E}}\frac{dv(s)}{ds} + \frac{d\theta}{ds}.$$

6. A simple proof of Gauss-Bonnet theorem. The formula

$$d\omega_{12} = -K\omega_1 \wedge \omega_2$$

is the key in apply Green's theorem to prove Gauss-Bonnet theorem. The formula can be re-written as

$$Kd\sigma = -d\omega_{12},$$

where $d\sigma = \omega_1 \wedge \omega_2$.

From the Gauss equation and Stoke's Theorem, the Gauss-Bonnet formula follows immediately for an oriented surface M with (piecewise smooth) boundary ∂M *on which we can globally define a moving frame*. That is, we can reprove the local Gauss-Bonnet formula quite effortlessly.

Proof: We start with an arbitrary moving frame $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$, and take a Darboux frame (i.e. a moving frame for the surface with $\mathbf{e}_1$ tangent to ∂M) $\bar{\mathbf{e}}_1, \bar{\mathbf{e}}_2, \bar{\mathbf{e}}_3$ along ∂M. We write $\bar{\mathbf{e}}_1 = \cos\theta \mathbf{e}_1 + \sin\theta \mathbf{e}_2, \bar{\mathbf{e}}_2 = -\sin\theta \mathbf{e}_1 + \cos\theta \mathbf{e}_2$ (where θ is smoothly chosen along the smooth pieces of ∂M and the exterior angle ϵ_j at P_j gives the jump of theta as we cross P_j). Then, by Stokes' theorem, we have

$$\int\int_M Kd\sigma = -\int\int_M d\omega_{12} = -\int_{\partial M} \omega_{12} = -\int_{\partial M} (\bar{\omega}_{12} - d\theta) = -\int_{\partial M} \kappa_g ds + (2\pi - \sum \epsilon_j).$$

7. Covariant Derivative, Connection Form

The covariant derivativ $D_{\mathbf{v}}\mathbf{e}_1$ is the tangential component of $d\mathbf{e}_1(\mathbf{v}) = \omega_{12}(\mathbf{v})\mathbf{e}_2 + \omega_{13}(\mathbf{v})\mathbf{e}_3$. Hence, $D_{\mathbf{v}}\mathbf{e}_1 = pr(d\mathbf{e}_1(\mathbf{v})) = \omega_{12}(\mathbf{v})\mathbf{e}_2$. Similarly, $D_{\mathbf{v}}\mathbf{e}_2 = \omega_{21}(\mathbf{v})\mathbf{e}_1$. The form ω_{12} is called the *connection form* and it measures the tangential twist of $\mathbf{e}_1$ and $\mathbf{e}_2$.

5 Appendix: Review of Surface Theory

We review here the theory of the surfaces we have learnt so far. Let M be a surface and $\boldsymbol{\sigma} : U \to M$ be an orthogonal parametrization (i.e. $F = 0$ in the first fundamental form). Recall that the vectors $\{\boldsymbol{\sigma}_u, \boldsymbol{\sigma}_v\}$ span $T_p(M)$. From $0 = F = \boldsymbol{\sigma}_u \cdot \boldsymbol{\sigma}_v$, we see that $\boldsymbol{\sigma}_u$ and $\boldsymbol{\sigma}_v$ are orthogonal. Let $\mathbf{e}_1 = \dfrac{\boldsymbol{\sigma}_u}{\sqrt{E}}$, $\mathbf{e}_2 = \dfrac{\sigma_v}{\sqrt{G}}$. Then $\{\mathbf{e}_1(p), \mathbf{e}_2(p)\}$ forms an orthonormal basis for $T_p(M), p \in M$ (i.e. $\|\mathbf{e}_1(p)\| = 1, \|\mathbf{e}_2(p)\| = 1$ and $\mathbf{e}_1(p) \cdot \mathbf{e}_2(p) = 0$). So $\{\mathbf{e}_1, \mathbf{e}_2\}$ is called a **moving frame** for the tangent spaces of S (since for each point $p \in M$, we have that $\{\mathbf{e}_1(p), \mathbf{e}_2(p)\}$ forms an orthonormal basis for $T_p(M)$, here the name "moving" because p varies from M and the name "frame" comes from the fact it is a basis). Such orthonormal basis exists becuase we can always apply the Gram-Schmidt orthnormalization procedure if $\boldsymbol{\sigma}$ is not an orthogonal parametrization. Let $\mathbf{e}_3 = \mathbf{n}$ be the unit normal(Gauss map). Then $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ forms an (moving) orthonormal basis for $\mathbf{R}^3$.

We have (for an orthogonal parametrization) (do your own calculations using the formulas in section 5.1), in below, E, F, G is first fundamental form with $F = 0$ and e, f, g are the second fundamental form.

$$(\mathbf{e}_1)_u = -\frac{E_v}{2\sqrt{EG}}\mathbf{e}_2 + \frac{e}{\sqrt{E}}\mathbf{e}_3,$$

$$(\mathbf{e}_1)_v = \frac{G_u}{2\sqrt{EG}}\mathbf{e}_2 + \frac{f}{\sqrt{E}}\mathbf{e}_3,$$

$$(\mathbf{e}_2)_u = \frac{E_v}{2\sqrt{EG}}\mathbf{e}_1 + \frac{f}{\sqrt{G}}\mathbf{e}_3,$$

$$(\mathbf{e}_2)_v = \frac{G_u}{2\sqrt{EG}}\mathbf{e}_1 + \frac{g}{\sqrt{G}}\mathbf{e}_3,$$

$$(\mathbf{e}_3)_u = -\frac{e}{\sqrt{E}}\mathbf{e}_1 - \frac{f}{\sqrt{G}}\mathbf{e}_2,$$

$$(\mathbf{e}_3)_v = -\frac{f}{\sqrt{E}}\mathbf{e}_1 - \frac{g}{\sqrt{G}}\mathbf{e}_2.$$

Gauss equation:

$$K = -\frac{1}{2\sqrt{EG}}\left(\frac{(\sqrt{E})_v}{\sqrt{G}}\right)_v + \left(\frac{(\sqrt{G})_u}{\sqrt{E}}\right)_u.$$

Codazzi equations:

$$\left(\frac{e}{\sqrt{E}}\right)_v - \left(\frac{f}{\sqrt{E}}\right)_u - g\frac{(\sqrt{E})_v}{G} - f\frac{(\sqrt{G})_u}{\sqrt{EG}} = 0,$$

$$\left(\frac{g}{\sqrt{G}}\right)_u - \left(\frac{f}{\sqrt{G}}\right)_v - e\frac{(\sqrt{G})_u}{E} - f\frac{(\sqrt{E})_v}{\sqrt{EG}} = 0.$$