Riemannian Geometry, Differential Operators on Riemannian Manifolds
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1 Differential operators

In this notes, we always assume that M is a compact oriented Riemannian manifold with the Levi-Civita connection $\bigtriangledown$, and $X, Y, ...$ are smooth vector fields.

- (p,q)-tensor is a smooth section of the bundle $T^p_qM = (\otimes^p TM) \otimes (\otimes^q T^*M)$.
- Contractions $c_{ij} : T^p_qM \to T^{p-1}_{q-1}$ defined by
  $c_{ij}(x_1 \otimes x_p \otimes y_1^* \otimes y_q^*) = y_j^*(x_i)x_1 \otimes \cdots \hat{x}_i \otimes \cdots \otimes x_p \otimes y_1^* \otimes \cdots \otimes \hat{y}_j^* \otimes y_q^*.$
- The interior product i_X of a vector field X with a covariant tensor of type $(0,p)$ is the tensor $c_{1,1}(X \otimes S)$ (of the type $(0, p-1)$), defined as
  $$(i_X(S))_m(x_1, \ldots, x_{p-1}) = S_m(X(m), x_1, \ldots, x_{p-1}).$$
- Extension of the covariant derivative: Let X be a vector field on X. The endomorphsim D_X of of $\Gamma(TM)$ has a unique extension of the endomorphism of the space of tensors, still denoted by D_X which is type-preserving and satisfies the following

  (1) For any $S \in \Gamma(T^p_q)$ ($p > 0, q > 0$ and any contraction c, $D_X(c(S)) = c(D_X S)$.

  (2) For any tensors S and T,
  $$D_X(S \otimes T) = D_X S \otimes T + S \otimes D_X T.$$
  If for example $S \in \Gamma(T^0_qM)$, the X_i are vector fields on M, then
  $$(D_X S)(X_1, \ldots, X_q) = X(S(X_1, \ldots, X_q)) - \sum_{i=1}^q S(X_1, \ldots, X_{i-1}, D_X X_i, \ldots, X_q).$$
  The readers should aware that, in general,
  $$(D_X S)(X_1, \ldots, X_n) \neq D_X(S(X_1, \ldots, X_n)).$$
- Hessian: If $f \in C^\infty(M)$, then the $(0,2)$-tensor Ddf is *symmetric*. The tensor Ddf is called the *Hessian* of f.

- Musical isomorphism: $\omega(Y) = g(X, Y)$ gives the map between X and ω. Flat $\flat$: the map $X \to \omega$, $\#$ is the inverse map.

- $div(X) := -tr(DX)$ and $\delta\alpha = -trD\alpha$ (the co-differential of α) where DX is the endormorphism $u \to D_u X$ and the $D\alpha$ is a covariant two-tensor, where the trace is computed with respect to the Riemannnian metric. Using the "musical isomorphism", these two notations can be viewed as equivalent. More general, for any $(0, q)$-tensor, we define the $div(T)$ as a $(0, q-1)$ type tensor as $\delta T = -tr_{12}DT$ where the notation $_{12}$ just means that the trace is taken with respect to the first two variables. It is easy to check

$$\delta(fT) = f\delta T - i_{\bigtriangledown f}T.$$

- The Laplacian of f is the function $\triangle f$ gievn by

$$\triangle f = div \bigtriangledown f = \delta df.$$

  The Euclidean Laplacian on $\mathbf{R}^n$ is given by

$$\triangle = -\sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}.$$

- **Bochner's formula**: For any smooth functions on (M, g), we have

$$g(\triangle(df), df) = |Ddf|^2 + \frac{1}{2} \triangle (|df|^2|) + Ric(\bigtriangledown f, \bigtriangledown f).$$

- **The divergence operator.** Let $X \in \Gamma(TM)$. Define

$$div(X) = tr\{Y \to \bigtriangledown_Y X\}.$$

  $div(X)$ is a smooth function M. It is called the *divergence of* X. The map $div : \Gamma(TM) \to C^\infty(M)$ given by $X \mapsto div(X)$ is called the *divergence operator*. In terms of local coordinate $(U; x^i)$, write

$$X|_U = X^i \frac{\partial}{\partial x^i},$$

then

$$div(X) = \frac{1}{\sqrt{G}} \sum_{i=1}^{m} \frac{\partial}{\partial x^i} \left(\sqrt{G} X^i \right),$$

where $G = det(g_{ij}), g_{ij} = g(\partial/\partial x^i, \partial/\partial x^j)$.

Proof. Let (U, x^i) be a local coordinate, and write

$$X|_U = X^i \frac{\partial}{\partial x^i}.$$

Then

$$\bigtriangledown X = \left(\frac{\partial X^i}{\partial x^j} + X^k \Gamma^i_{kj} \right) dx^j \otimes \frac{\partial}{\partial x^i}.$$

Hence

$$div(X) = \sum_{i=1}^{m} \left(\frac{\partial X^i}{\partial x^i} + X^k \Gamma^i_{ki} \right).$$

So we see that the div operator is **a differential operator of first order** acting on X.

By the formula,

$$\Gamma^i_{ki} = \frac{1}{2} g^{ij} \frac{\partial g_{ij}}{\partial x^k} = \frac{1}{2G}\frac{\partial G}{\partial x^k} = \frac{1}{\sqrt{G}}\frac{\partial \sqrt{G}}{\partial x^k}.$$

Thus

$$div(X) = \frac{\partial X^i}{\partial x^i} + \frac{X^k}{\sqrt{G}}\frac{\partial \sqrt{G}}{\partial x^k} = \frac{1}{\sqrt{G}}\frac{\partial}{\partial x^i}\left(\sqrt{G} X^i\right).$$

- **The gradient of** f. Let $f \in C^\infty(M)$, define a tangent vector field $grad(f) \in \Gamma(TM)$, by

  $$g\left(grad(f), X\right) = df(X) = X(f),$$

  for every smooth tangent vector field X. The tangent vector field $grad(f)$ is called the *gradient* of f. In terms of local coordinate $(U; x^i)$,

  $$grad(f) = \sum_{j=1}^m \left(\sum_{i=1}^m g^{ij} \frac{\partial f}{\partial x^i} \right) \frac{\partial}{\partial x^j},$$

  where $g_{ij} = g(\partial/\partial x^i, \partial/\partial x^j), (g^{ij}) = (g_{ij})^{-1}$. Again, the "grad operator" $grad : C^\infty(M) \to \Gamma(TM)$ defined by $f \mapsto grad(f)$ is **a differential operator of first order** acting on f.

- **Beltrami-Laplace operator**. Let $f \in C^\infty(M)$, define $\triangle f = div(grad(f))$. It is called the *Beltrami-Laplace operator.* The operator $\triangle f = div \circ grad : C^\infty(M) \to C^\infty(M)$ is a very important differential operator.

  In local coordinate $(U; x^i)$,

  $$\triangle f = \frac{1}{\sqrt{G}} \sum_{i=1}^m \frac{\partial}{\partial x^i} \left(\sum_{j=1}^m \sqrt{G} g^{ij} \frac{\partial f}{\partial x^j} \right),$$

  where $G = det(g_{ij}), g_{ij} = g(\partial/\partial x^i, \partial/\partial x^j), (g^{ij}) = (g_{ij})^{-1}$.

- **Volume form, interior product**. In a local coordinate $(U; x^i)$, let

  $$\eta = \sqrt{G} dx^1 \wedge \cdots \wedge dx^m.$$

η in fact is a global m-form, called *the volume form of* M. Fix a smooth tangent vector field X, the interior product $i(X)$ is defined by, for every tangent vector fields $X_1, \ldots, X_{m-1}$,

$$(i(X)\eta)(X_1, \ldots, X_{m-1}) = \eta(X, X_1, \ldots, X_{m-1}).$$

Then we have, for every smooth tangent vector field X,

$$d(i(X)\eta) = div(X)\eta.$$

Proof: By definition, $\eta = \sqrt{G}dx^1 \wedge \cdots \wedge dx^m$. Hence, from the forum la above,

$$\begin{aligned} div(X)\eta &= \frac{\partial}{\partial x^i}\left(\sqrt{G}X^i\right) dx^1 \wedge \cdots \wedge dx^m \\ &= \sum_{i=1}^m d((-1)^{i+1}\sqrt{G}X^i) \wedge dx^1 \wedge \cdots \wedge d\hat{x}^i \wedge \cdots \wedge dx^m. \end{aligned}$$

Write

$$\omega = \sum_{i=1}^m (-1)^{i+1}\sqrt{G}X^i dx^1 \wedge \cdots \wedge d\hat{x}^i \wedge \cdots \wedge dx^m.$$

It is easy to verify that ω is independent of the choice of coordinates. So ω is a globally defined $(m-1)$-form on M. The above identity gives

$$div(X)\eta = d\omega.$$

It is easy to verify that $\omega = i(X)\eta$ by definition.

- **The divergence theorem**. Use the above identity that $div(X)\eta = d\omega$. and Stokes theorem, we get the divergence theorem: *Let* (M, g) *be a compact oriented Riemannian manifold, then, for every smooth tangent vector field* X,

  $$\int (divX)\eta = 0,$$

  where η *is the volume form.*

- **Global inner product for differential forms** We first define the inner product for differential forms. Let ϕ, ψ are two r-forms. Let (U, x^i) be a local coordinate. We write

$$\phi|_U = \frac{1}{r!}\phi_{i_1\cdots i_r} dx^{i_1} \wedge \cdots \wedge dx^{i_r},$$

$$\psi|_U = \frac{1}{r!}\psi_{j_1\cdots j_r} dx^{j_1} \wedge \cdots \wedge dx^{j_r}.$$

  We define, the inner product $< \ , \ >$ of ϕ, ψ as

$$< \phi, \psi >= \frac{1}{r!}\phi^{i_1\cdots i_r}\psi_{i_1\cdots i_r} = \sum_{i_1<\cdots<i_r} \phi^{i_1\cdots i_r}\psi_{i_1\cdots i_r},$$

  where $\phi^{i_1\cdots i_r} = g^{i_1 j_1}\cdots g^{i_r j_r}\phi_{j_1\cdots j_r}$. It is important to note that the definition is independent of the choice of local coordinates. We also have $< \phi, \phi >\geq 0$ and $< \phi, \phi >= 0$ if and only if $\phi = 0$.

  We now define the **global** inner product of ϕ, ψ as

$$(\phi, \psi) = \int_M < \phi, \psi > \eta,$$

  where η is the volume form of M.

- **The exterior differential operator d and its co-operator** Denote by $\Lambda^r(M)$ the set of smooth r-forms on M. Let $(\ , \ )$ be the (global) inner product defined above. As the formal adjoint operator of the exterior differential operator d, the *codifferential operator* $\delta : \Lambda^{r+1}(M) \to \Lambda^r(M)$ is defined by, for every $\phi \in \Lambda^r(M), \psi \in \Lambda^{r+1}(M)$,

$$(d\phi, \psi) = (\phi, \delta\psi).$$

- **Hodge-star operator**. In order to find the expression of the codifferential operator δ, we introduce the Hodge-star operator $*$, which is an isomorphism $* : \Lambda^r(M) \to \Lambda^{m-r}(M)$ defined by, for every $\phi, \eta \in \Lambda^r(M)$,

$$\phi \wedge (*\psi) =< \phi, \psi > \eta.$$

Let ω be a r-form. Let (U, x^i) be a local coordinate. We write

$$\omega|_U = \frac{1}{r!} \sum_{i_1,\ldots,i_r} a_{i_1\cdots i_r} dx^{i_1} \wedge \cdots \wedge dx^{i_r}.$$

Then

$$*\omega = \frac{\sqrt{G}}{r!(m-r)!} \delta^{1\cdots m}_{i_1\cdots i_m} a^{i_1\cdots i_r} dx^{i_{r+1}} \wedge \cdots \wedge dx^{i_m},$$

where

$$a^{i_1\cdots i_r} = g^{i_1 j_1} \cdots g^{i_r j_r} a_{j_1\cdots j_r},$$

and $\delta^{1\cdots m}_{i_1\cdots i_m}$ is the Levi-Civita permutation symbol, i.e. $\delta^{12\cdots m}_{i_1\cdots i_m} = 1$ if $(i_1 \cdots i_m)$ is an even permutation of $(12\ldots m)$, $\delta^{12\cdots m}_{i_1\cdots i_m} = -1$ if $(i_1 \cdots i_m)$ is an odd permutation of $(12\ldots m)$, $\delta^{12\cdots m}_{i_1\cdots i_m} = 0$ otherwise. It can be shown that $*\omega$ is is independent of the choice of local coordinates. So $*\omega$ is a globally defined $(m-r)$-form (it can be regarded as an alternative definition). The operator $*$ which sends r-forms to $(m-r)$-forms.

It has the following properties, for any r-forms ϕ and ψ:

(1) $\phi \wedge *\psi =< \phi, \psi > \eta$,

(2) $*\eta = 1, *1 = \eta$,

(3) $*(*\phi) = (-1)^{r(m+1)}\phi$,

(4) $(*\phi, *\psi) = (\phi, \psi)$.

- **Expression of the codifferential operator δ in terms of the Hodge-Star operator**. Define $\delta = (-1)^{mr+1} * \circ d \circ * : \Lambda^{r+1}(M) \to \Lambda^r(M)$, where $\Lambda^r(M)$ is the set of smooth r-forms, is called the *codifferential operator*. It is easy to verify that $\delta \circ \delta = 0$. We also have the following very important property for δ: For $\phi \in \Lambda^r(M), \psi \in \Lambda^{r+1}(M)$, we have

  $$(d\phi, \psi) = (\phi, \delta\psi),$$

  i.e. δ **is conjugate to** d. So $(-1)^{mr+1} * \circ d \circ *$ is the expression of the odifferential operator δ.

Proof. Note

$$\begin{aligned} d(\phi \wedge *\psi) &= d\phi \wedge *\psi + (-1)^r \phi \wedge d(*\psi) \\ &= d\phi \wedge *\psi + (-1)^r (-1)^{mr+r} \phi \wedge *(*d*\psi) \\ &= d\phi \wedge *\psi - \phi \wedge *\delta\psi. \end{aligned}$$

Then desired identity is obtained by applying the Stokes theorem.

- **Hodge-Laplace operator.** We define the Hodge-Laplace operator

$$\tilde{\triangle} = d\delta + \delta d : \Lambda^r(M) \to \Lambda^r(M).$$

For $f \in C^\infty(M)$, then $\delta(f) = 0$, so

$$\tilde{\triangle}(f) = \delta(df) = -*d*df, \quad \tilde{\triangle} f\eta = *\tilde{\triangle} f = -d*df.$$

Let (U, x^i) be a local coordinate, then

$$df|_U = \frac{\partial f}{\partial x^i} dx^i,$$

$$\begin{aligned} *df|_U &= \frac{\sqrt{G}}{(m-1)!} \delta^{1\cdots m}_{i_1\cdots i_m} g^{i_1 j} \frac{\partial f}{\partial x^j} dx^{i_2} \wedge \cdots \wedge dx^{i_m} \\ &= \sqrt{G} \sum_{i=1}^m (-1)^{i+1} g^{ij} \frac{\partial f}{\partial x^j} dx^1 \wedge \cdots \wedge d\hat{x}^i \wedge \cdots \wedge dx^m. \end{aligned}$$

Hence

$$\begin{aligned} (\tilde{\triangle} f)\eta|_U &= -d(*df)|_U = -\frac{\partial}{\partial x^i}\left(\sqrt{G} g^{ij} \frac{\partial f}{\partial x^j}\right) dx^1 \wedge \cdots \wedge dx^m \\ &= -\triangle f \eta|_U. \end{aligned}$$

This tells us

$$\tilde{\triangle} f = -\triangle f.$$

So $-\tilde{\triangle}$ when acts on $C^\infty(M)$ is the Beltrami-Laplace operator $\triangle$.

- **Hodge Theory**. In this section, we denote the Hodge-Laplace operator by $\triangle$. Let $\mathcal{H}^r(M) = ker\triangle$ and $\mathcal{H} = \bigoplus \mathcal{H}^r(M)$. Let $\bigwedge^*(M) = \bigoplus_{r=0}^{\infty} \Lambda^r(M)$.

  The Hodge theorem *Let (M, g) be an n-dimensional compact oriented Riemannian manifold without boundary. For each integer $0 \leq r \leq n$, $\mathcal{H}^r(M)$ is finite dimensional, and there exists a bounded linear operator $G : \bigwedge^*(M) \to \bigwedge^*(M)$ (called Green's operator) such that*

  (a) $kerG = \mathcal{H}$;

  (b) G keeps types, and commute with the operators $*, d$ and δ;

  (c) G is a compact operator, i.e. the closure of image of an arbitrary bounded subset of $\bigwedge^*(M)$ under G is compact;

  (d) $I = \mathcal{H} + \triangle \circ G$, where I is the identity operator, and $\mathcal{H}$ is the orthogonal projection from $\bigwedge^*(M)$ to $\mathcal{H}$ with respect to the inner product $(\ , \ )$.

  From the Hodge theorem, since $I = \mathcal{H} + \triangle \circ G$, we can write (called the Hodge-decomposition)

  Corollary(Hodge-decomposition)

$$
\begin{aligned}
\Lambda^r(M) &= \triangle(\Lambda^r(M)) \oplus \mathcal{H}^r(M) \\
&= d\delta\Lambda^r(M) \oplus \delta d\Lambda^r(M) \oplus \mathcal{H}^r(M) \\
&= d\Lambda^{r-1}(M) \oplus \delta\Lambda^{r+1}(M) \oplus \mathcal{H}^r(M).
\end{aligned}
$$