§5.2 The Levi-Civita Connection on Surfaces

In this section, we define the parallel transport of vector fields on a surface M, and then we introduce the concept of the Levi-Civita connection, which is also an "extrinsic" quantity depends on the first fundamental form only (it then can be generalized to any higher-dimensional Riemannian manifolds).

1 Parallel transport of vector fields on a surface M

Recall that in $\mathbf{R}^2$ (plane) or $\mathbf{R}^3$ (Euclidean space), we can free to move the vectors, i.e. as long as we keep its norm and direction, the vectors are going to be the same (i.e. it doesn't matter where we put the vector). This is a very important fact, since using the parallel transport, we can perform linear operations for vectors, such as addition and subtraction etc., and then use the limit to do the differential operations. However, given a surface M, if we perform the parallel transport for a tangent vector $\mathbf{v}_p \in T_p(M)$ in the same sense as in $\mathbf{R}^3$ (i.e. moving by keeping the norm and direction), then the resulting vector may not stay in the tangent space any more (which becomes "extrinsic") if the surface M is not flat.

To overcome this difficulty, we need to do some modification. First, we introduce the concept of the vector fields.

Definition 5.2.1 *A vector field* $\mathbf{X}$ *over* M *assigns, at every point* $p \in M$*, the tangent vector* $\mathbf{X}_p \in T_p(M)$.

Definition 5.2.2 *Let* $C : \boldsymbol{\alpha} = \boldsymbol{\alpha}(t), t \in (a, b)$ *be a curve in* M*. A differentiable (vector-valued) function* $\mathbf{X} : (a, b) \to \mathbf{R}^3$ *with the property that* $\mathbf{X}(t) \in T_{\boldsymbol{\alpha}(t)}(M)$*, is called a vector field on* M *along the curve* C.

Example 5.2.1. Let $C : \boldsymbol{\alpha}(t) = (\cos t, \sin t, 0)$ on the unit sphere S^2" $x^2 + y^2 + z^2 = 1$ (so we can take $\mathbf{n}(t) = \boldsymbol{\alpha}(t)$ as the unit-normal to S^2 along the curve C). Let

$$\mathbf{X}(t) = (\sin t, -\cos t, t^2).$$

Then, since $\mathbf{n}(t) \cdot \mathbf{X}(t) = 0$, $\mathbf{X}(t)$ is a vector field on S^2 along the curve C. It can also be easily checked that

$$\mathbf{T}(t) = (-e^t \sin t, e^t \cos t, e^{2t})$$

is also a vector field on S^2 along the curve C. Obviously,

Lemma 5.2.1 *Let* $\boldsymbol{\sigma} : U \to \mathbf{R}^3$ *be a local parametrization of* M*. Let* $C : \boldsymbol{\alpha}(t) = \boldsymbol{\sigma}(u(t), v(t))$*. Then every vector field on* M *along the curve* C *can be written as*

$$\mathbf{X}(t) = X^1(t)\boldsymbol{\sigma}_u(u(t), v(t)) + X^2(t)\boldsymbol{\sigma}_v(u(t), v(t))$$

where $X^1(t), X^2(t)$ *are differentiable function.*

In the example 5.2.1, if we consider

$$\frac{d\mathbf{X}}{dt} = (\cos t, \sin t, 2t)$$

and

$$\frac{d\mathbf{T}}{dt} = (-e^t(\sin t + \cos t), e^t(\cos t - \sin t), 2e^{2t})),$$

then they both failed to be the vector fields along the curve C.

In general, let $\mathbf{X} = \mathbf{X}(t)$ be a vector field along the curve C, write

$$\mathbf{X}(t) = \sum_{i=1}^{2} X^i(t)\boldsymbol{\sigma}_i(u_1(t), u_2(t)),$$

where, we denote $\boldsymbol{\sigma}_1 = \boldsymbol{\sigma}_u, \boldsymbol{\sigma}_2 = bsigma_v$, $u_1 = u, u_2 = v$, then, using (5.1.2)-(5.1.4) in section 5.1, we have

$$\frac{d\mathbf{X}}{dt} = \sum_i \frac{dX^i}{dt}\boldsymbol{\sigma}_i + \sum_{i.j} X^i \boldsymbol{\sigma}_{ij}\frac{du^j}{dt}$$

$$= \sum_k \left(\frac{dX^k}{dt} + \sum_{i,j} \Gamma^k_{ij} X^i \frac{du^j}{dt} \right) \boldsymbol{\sigma}_k + \sum_{i,j} h_{ij} X^i \frac{du^j}{dt}\mathbf{n}, \tag{5.2.1}$$

where $h_{11} = e, h_{12} = h_{21} = f, h_{22} = g$, the second fundamental forms of M. The above expression gives a decomposition of the vector $\frac{d\mathbf{X}}{dt}$ in the tangent and normal directions. We define

$$\frac{\bigtriangledown \mathbf{X}}{dt} = \sum_k \left(\frac{dX^k}{dt} + \sum_{i,j} \Gamma^k_{ij} X^i \frac{du^j}{dt} \right) \boldsymbol{\sigma}_k, \tag{5.2.2}$$

which is called the **covariant derivative of the vector field $\mathbf{X}(t)$ along the curve** C. Note that, since $\mathbf{X}$ and its derivative $d\mathbf{X}/dt$ are independent of the choice of the parametrization $\boldsymbol{\sigma}$, and $\frac{\triangledown \mathbf{X}}{dt}$ is the projection of $d\mathbf{X}/dt$ to the tangent plane of M, it is also independent of the choice of the parametrization $\boldsymbol{\sigma}$. It is easy to check that, for two vector fields,

$$\frac{\triangledown(\mathbf{X}+\mathbf{Y})}{dt} = \frac{\triangledown \mathbf{X}}{dt} + \frac{\triangledown \mathbf{Y}}{dt}, \quad \frac{\triangledown(\lambda \mathbf{X})}{dt} = \lambda \frac{\triangledown \mathbf{X}}{dt},$$

where λ is a constant.

Definition 5.2.3 *If* $\frac{\triangledown \mathbf{X}}{dt} \equiv 0$, *then we say that* $\mathbf{X}$ *is a parallel vector field along the curve* C.

When we write

$$\mathbf{X}(t) = \sum_{i=1}^{2} X^i(t)\boldsymbol{\sigma}_i(u_1(t), u_2(t)),$$

then the parallel vector field $\mathbf{X}$ is determined by the following differential equations

$$\frac{dX^k}{dt} + \sum_{i,j} \Gamma^k_{ij} X^i \frac{du^j}{dt} = 0, \quad k = 1, 2. \tag{5.2.2}$$

Obviously (see the geodesic equations in section 4.2), a curve $\boldsymbol{\alpha}$ on M is geodesic if and only if the vector field $\boldsymbol{\alpha}'(t)$ is a parallel vector field along itself.

We have the following important properties (it can be easily verified).

Proposition 5.2.1 *Let* $\mathbf{X}(t), \mathbf{Y}(t)$ *be two vector field along a curve* C. *Then*

(1)

$$\frac{d}{dt}(\mathbf{X}(t) \cdot \mathbf{Y}(t)) = \frac{\triangledown \mathbf{X}}{dt} \cdot \mathbf{Y}(t) + \mathbf{X}(t) \cdot \frac{\triangledown \mathbf{Y}}{dt},$$

(2) Hence if $\mathbf{X}(t)$ *and* $\mathbf{Y}(t)$ *are parallel along a curve* C. *Then* $\|\mathbf{X}(t)\|, \|\mathbf{Y}(t)\|$, $\mathbf{X}(t) \cdot \mathbf{Y}(t)$ *are all constant. Thus, the angle between two parallel vector fields along a curve* C *is also constant.*

Proposition 5.2.1 provided some geometric meaning of what does ”parallel” of two vector fields mean.

Let $\mathbf{X}$ be a parallel vector field along a curve C. Take two point $\boldsymbol{\sigma}(t_1)$ and $\boldsymbol{\sigma}(t_2)$ on C, we call that vector $\mathbf{X}(t_2)$ is obtained by **parallel transport** of $\mathbf{X}(t_1)$ along the curve C. This kind of "parallel transport" is called called the **Levi-Civita parallel transport**. It defines a map

$$\mathcal{P}^{\boldsymbol{\sigma}(t)} : T_{\boldsymbol{\sigma}(t_0)}(M) \to T_{\boldsymbol{\sigma}(t)}(M)$$

given by $\mathcal{P}^{\boldsymbol{\sigma}(t)}(\mathbf{X}(t_0) = \mathbf{X}(t)$ where $\mathbf{X}$ is the parallel vector field along a curve C with the initial value $\mathbf{X}(t_0)$ (such vector filed also exists since we can always solve the differential equations in (5.2.3). The map $\mathcal{P}^{\boldsymbol{\sigma}(t)}$ provides a isometry since it keeps the dot product on the tangent spaces.

Please click here to see more details about the meaning of the parallel transport.

2 The Levi-Civita Connection on M

Let M be a surface in $\mathbf{R}^3$, and let $\mathbf{X}$ be a vector field on M, let f be a smooth functio non M. Recall that we have defined $D_{\mathbf{X_p}}(f)(p)$, the directional derivative of f at p with respect to the direction $\mathbf{X}_p$ (see (3.4.1) for the definition). In particular, Let $\boldsymbol{\sigma} : U \subset \mathbf{R}^2 \to \mathbf{R}^3$ be a parametrization of M. Then $D_{\boldsymbol{\sigma}_u} f = \frac{\partial f}{\partial u}$, where we regard $f(u,v) = f \circ \boldsymbol{\sigma}(u,v)$.

Now, let $\boldsymbol{\sigma}(u_1, u_2) : U \subset \mathbf{R}^2 \to \mathbf{R}^3$ be a parametrization of M, for given two vector fields

$$\mathbf{X} = X^1\boldsymbol{\sigma}_1 + X^2\boldsymbol{\sigma}_2, \quad \mathbf{Y} = Y^1\boldsymbol{\sigma}_1 + Y^2\boldsymbol{\sigma}_2,$$

where $\boldsymbol{\sigma}_1 = \boldsymbol{\sigma}_u, \boldsymbol{\sigma}_2 = bsigma_v$, we define the action of $\mathbf{Y}$ on $\mathbf{X}$ as

$$\mathbf{Y}(\mathbf{X}) = \sum_i X^i \frac{\partial \mathbf{Y}}{\partial u^i} = \sum_{i,j} X^i \left(\frac{\partial Y^j}{\partial u^i} \boldsymbol{\sigma}_j + Y^j \boldsymbol{\sigma}_{ji} \right).$$

Similar to the (5.2.1) of the definition of $\bigtriangledown\mathbf{Y}/dt$, we have the following decomposition in the direction of tangent and normal direction:

$$\mathbf{Y}(\mathbf{X}) = \sum_{i,j} X^i \frac{\partial Y^j}{\partial u^i} \boldsymbol{\sigma}_j + \sum_{i,j} X^i Y^j \boldsymbol{\sigma}_{ji}$$

$$= \sum_{i,k} \left(X^i \frac{\partial Y^k}{\partial u^i} + \sum_j X^i Y^j \Gamma^k_{ij} \right) \boldsymbol{\sigma}_k + \sum_{i,j} X^i Y^j h_{ij} \mathbf{n}.$$

This motivates the following definition

Definition 5.2.4*Let* $\mathbf{X} = X^1\boldsymbol{\sigma}_1 + X^2\boldsymbol{\sigma}_2, \mathbf{Y} = Y^1\boldsymbol{\sigma}_1 + Y^2\boldsymbol{\sigma}_2$ *be two vector fields on* M*, we define*

$$\bigtriangledown_{\mathbf{X}}\mathbf{Y} = \sum_{i,k}\left(X^i\frac{\partial Y^k}{\partial u^i} + \sum_j X^iY^j\Gamma^k_{ij}\right)\boldsymbol{\sigma}_k$$

which is called the Levi-civita connection.

Note that $\bigtriangledown_{\mathbf{X}}\mathbf{Y}$ is just the orthogonal projection of $\mathbf{X}(\mathbf{Y})$ to the tangent spaces of M. Hence it is an intrisic property. It provides a way to differentiate the vector field $\mathbf{Y}$ in the direction $\mathbf{X}$ to **get back** to a vector field again (note that $\mathbf{X}(\mathbf{Y})$ is not, in general, a vector field anymore). From the definition, we have

$$\bigtriangledown_{\boldsymbol{\sigma}_i}\boldsymbol{\sigma}_j = \sum_k \Gamma^k_{ij}\boldsymbol{\sigma}_k.$$

From section 5.1, we can verify that

$$\Gamma^k_{ij} = \frac{1}{2}\sum_{l=1}^{2} g^{kl}(\partial_i g_{lj} - \partial_j g_{li} - \partial_l g_{ij}$$

where $g_{11} = E, g_{12} = g_{21} = F, g_{22} = G$ is the second fundamental form, and (g^{kl}) is the inverse matrix of (g_{ij}).

We define the Lie-bracket of two vector fields $\mathbf{X}, \mathbf{Y}$ as

$$[\mathbf{X}, \mathbf{Y}] = \mathbf{X}(\mathbf{Y}) - \mathbf{Y}(\mathbf{X}).$$

We define the curvature operator as

$$R(\mathbf{X}, \mathbf{Y})\mathbf{Z} = \bigtriangledown_{\mathbf{X}}\bigtriangledown_{\mathbf{Y}}\mathbf{Z} - \bigtriangledown_{\mathbf{Y}}\bigtriangledown_{\mathbf{X}}\mathbf{Z} - \bigtriangledown_{[\mathbf{X},\mathbf{Y}]}\mathbf{Z}.$$

Then it can be checked (it takes time, but it is doable. If you are interested it, you can try to derive it) that the Gauss curvature can be expressed as

$$K = -\frac{(R(\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2)\boldsymbol{\sigma}_1)\cdot\boldsymbol{\sigma}_2}{\|\boldsymbol{\sigma}_1\|^2\|\boldsymbol{\sigma}_2\|^2 - (\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2)^2}.$$