Riemannian Geometry: Overview

by Min Ru, University of Houston

1 Overview: What is the course about?

Consider a submanifold of Euclidean space, e.g. a curve or a surface in $\mathbf{R}^n$. On the submanifold lives creatures that are unaware of the surrounding space. They can measure the distance on the submanifold but not distances in space. How much of a geometry of the submanifold can they find out?

In the early 1800s Gauss asked how much of the geometry of a surface is independent of how it bends in space, i.e. how much of the geometry that remains the same if we perform an isometry that doesn't change the distance between points. Riemannian geometry is designed to describe the universe of creatures who live on a curved surface and who unaware of the surrounding space outside and can only measure distances and areas on the surface. This is called the intrinsic geometry. This lead to the modern notion of a manifold independent of a surrounding space. This course is about manifolds with a notion of a distance, how this can be used to define curvatures and how the curvatures can characterize a manifold. Einstein realized how this theory could be used to describe how space curves under the influence of gravity which lead to the general theory of relativity.

The core topics are:

- Review of curves and surfaces in $\mathbf{R}^3$. Abstract 2-dimensional Riemannian manifolds. Theory of submanifolds of $\mathbf{R}^n$ (Euclidean space).
- Abstract manifolds, tangent space, vector bundles.
- Riemannian metric on a manifold. The Levi-Civita connection. Connections on vector bundles.

- Geodesics and the exponential map.
- The curvature tensor.
- Curvature and local geometry. Jacobi fields.
- Cartan's formulism(Moving frames and the structure equations).
- Harmonic forms and Hodge Theory.

texbook: I will use my own notes. Here are some useful on-line sites for lecture notes: http://www.matematik.lu.se/matematiklu/personal/sigma/Riemann.pdf, and
http://www.math.ucsd.edu/ lindblad/250b/250b.html.