Answers to Homework 1, Math 4364, Fall 2009

2.1

3a. Let $f(x) = x^3 - 7x^2 + 14x - 6$. Then $f(0) = -6,\ f(1) == 2$.

$$a_1 = a = 0,\ b_1 = b = 1, P_1 = \frac{a_1 + b_1}{2} = 0.5,\ \ f(P_1) \approx -0.625,$$

$$a_2 = 0.5, b_2 = 1, P_2 = \frac{a_2 + b_2}{2} = 0.75, f(P_2) \approx 0.98437,$$

$$a_3 = 0.5, b_3 = 0.75, P_3 = \frac{a_3 + b_3}{2} = 0.625, f(P_3) \approx 0.25976$$

$$a_4 = 0.5, b_4 = 0.625, P_4 = \frac{a_4 + b_4}{2} = 0.5625, f(P_4) \approx -0.16186$$

$$a_5 = 0.5625, b_5 = 0.625, P_5 = \frac{a_5 + b_5}{2} = 0.59375, f(P_5) \approx 0.054046$$

$$a_6 = 0.5, b_6 = 0.59375, P_6 = \frac{a_6 + b_6}{2} = 0.578125, f(P_6) \approx -0.052624$$

$$a_7 = 0578125, b_7 = 0.59375, P_7 = \frac{a_7 + b_7}{2} = 0.5859375, f(P_7) \approx 0.001031$$

So we obtain the solution $P_7 = 0.5859375$ with $\left|\frac{b_7 - a_7}{2}\right| < 10^{-2}$.

5b.

I	P	F(P)
1	5.00000000e-01	8.9872127e-01
2	2.50000000e-01	-2.8474583e-02
3	3.75000000e-01	4.3936641e-01
4	3.12500000e-01	2.0668169e-01
5	2.81250000e-01	8.9433196e-02
6	2.65625000e-01	3.0564234e-02
7	2.57812500e-01	1.0663677e-03
8	2.53906250e-01	-1.3698684e-02
9	2.55859375e-01	-6.3148068e-03
10	2.56835938e-01	-2.6238823e-03

11	2.57324219e-01	-7.7867310e-04
12	2.57568359e-01	1.4386834e-04
13	2.57446289e-01	-3.1739712e-04
14	2.57507324e-01	-8.6763073e-05
15	2.57537842e-01	2.8552963e-05
16	2.57522583e-01	-2.9104973e-05
17	2.57530212e-01	-2.7598471e-07

So the solution $P_{17} = 0.2575302$ accurate to within 10^{-5}.

15. $|P_n - P| \le 2^{-n}(b-a) < 10^{-4}$,

i.e. $-n\log_{10} 2 < -4 \Rightarrow n > \dfrac{4}{\log_{10} 2} \approx 13.29$.

So $n \ge 14$.

And $P_{14} = 1.32476807$.

2.2

1a. $x = (3 + x - 2x^2)^{1/4} \Leftrightarrow x^4 = 3 + x - 2x^2 \Leftrightarrow f(x) = 0$.

1d. $x = \dfrac{3x^4 + 2x^2 + 3}{4x^3 + 4x - 1} \Leftrightarrow 4x^4 + 4x^2 - x = 3x^4 + 2x^2 + 3 \Leftrightarrow f(x) = 0$.

5 (i). $g(x) = (3x^2 + 3)^{1/4}$ and $P_0 = 1$

$P_1 = 1.56508$;

$P_2 = 1.79357$;

$P_3 = 1.88594$;

$P_4 = 1.92285$;

$P_5 = 1.93751$;

$P_6 = 1.94332$;

The error $|P_6 - P_5| = 0.00581 < 10^{-2}$.

(ii) Another "good" function $g(x) = x - \dfrac{x^4 - 3x^2 - 3}{4x^3 - 6x}$ and $P_0 = 2$

$P_1 = 1.95$;

$P_2 = 1.947132$;

The error $|P_2 - P_1| = 0.002868 < 10^{-2}$.

11. Consider an interval $[a,b]$, where $a > 0$.

Let $g(x) = \dfrac{5}{x^2} + 2$, then

$|g'(x)| = \left|\dfrac{10}{x^3}\right| \le \dfrac{10}{a^3} < 1 \;\Rightarrow\; a > 10^{1/3} \approx 2.15443469$. Take $a = 2.5$. ($g'(x) = \frac{10}{x^3}$ is a decreasing function for $x > 0$.)

$a = 2.5 \le 2 + \dfrac{5}{b^2} \le 2 + \dfrac{5}{a^2} \le b \;\Rightarrow\; 2.8 \le b \le 3.162$. Take $b = 3$. ($g(x) = \frac{5}{x^2} + 2$ is a decreasing function for $x > 0$.)

So the interval is $[2.5,3]$.

(1) Let $P_0 = 2.5$. $k = 0.64$ ($|g'(x)| = \left|\dfrac{10}{x^3}\right| \le k < 1$, for all $x \in [2.5,3]$)

Then

$|P_n - P| \le k^n \max\{P_0 - a, b - P_0\} = 0.64^n * 0.5 < 10^{-5} \;\Rightarrow n > 24.24$.

So $n \ge 25$.

(2)

I	P
1	2.80000000e+00
2	2.63775510e+00
3	2.71862291e+00
4	2.67650663e+00
5	2.69796454e+00
6	2.68690635e+00
7	2.69257202e+00
8	2.68966049e+00
9	2.69115440e+00
10	2.69038727e+00
11	2.69078104e+00
12	2.69057887e+00
13	2.69068265e+00
14	2.69062937e+00
15	2.69065673e+00
16	2.69064268e+00
17	2.69064989e+00