Answers to Homework 4, Math 4363 (Fall 2009)

6.2

9a.

$$m_{21} = \frac{5.31}{0.03} = 177\,,$$

$$x_2 = \frac{47.0 - 59.2 \times 177}{-6.10 - 58.9 \times 177} \approx \frac{47.0 - 10400}{-6.10 - 10400} \approx \frac{-10300}{-10400} \approx 0.990\,,$$

$$x_1 \approx \frac{59.2 - 58.9 \times 0.990}{0.03} \approx \frac{59.2 - 58.3}{0.03} \approx 30.0$$

11a.

$$m_{21} = \frac{5.31}{0.03} = 177\,,$$

$$x_2 = \frac{47.0 - 59.2 \times 177}{-6.10 - 58.9 \times 177} \approx \frac{47.0 - 10500}{-6.10 - 10400} \approx \frac{-10500}{-10400} \approx 1.01\,,$$

$$x_1 \approx \frac{59.2 - 58.9 \times 1.01}{0.03} \approx \frac{59.2 - 59.5}{0.03} \approx -10.0\,.$$

13a

$$m_{21} = \frac{0.03}{5.31} \approx 0.00564,\,,$$

$$x_2 = \frac{59.2 - 47.0 \times 0.00564}{58.9 + 6.10 \times 0.00564} \approx \frac{59.2 - 0.265}{58.9 + 0.0344} \approx \frac{58.9}{58.9} \approx 1.00\,,$$

$$x_1 \approx \frac{47.0 + 6.10 \times 1.00}{5.31} \approx \frac{53.1}{5.31} \approx 10.0\,.$$

15a

$$m_{21} = \frac{0.03}{5.31} \approx 0.00565,\,,$$

$$x_2 = \frac{59.2 - 47.0 \times 0.00565}{58.9 + 6.10 \times 0.00565} \approx \frac{59.2 - 0.266}{58.9 + 0.0345} \approx \frac{58.9}{58.9} \approx 1.00\,,$$

$$x_1 \approx \frac{47.0 + 6.10 \times 1.00}{5.31} \approx \frac{53.1}{5.31} \approx 10.0\,.$$

6.5

5a.

$$A = \begin{pmatrix} 2 & -1 & 1 \\ 3 & 3 & 9 \\ 3 & 3 & 5 \end{pmatrix},$$

$$M_1 = \begin{pmatrix} 1 & 0 & 0 \\ -1.5 & 1 & 0 \\ -1.5 & 0 & 1 \end{pmatrix}, \qquad M_1 A = \begin{pmatrix} 2 & -1 & 1 \\ 0 & 4.5 & 7.5 \\ 0 & 4.5 & 3.5 \end{pmatrix};$$

$$M_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}, \qquad M_2 M_1 A = \begin{pmatrix} 2 & -1 & 1 \\ 0 & 4.5 & 7.5 \\ 0 & 4.5 & 3.5 \end{pmatrix} = \begin{pmatrix} 2 & -1 & 1 \\ 0 & 4.5 & 7.5 \\ 0 & 0 & -4 \end{pmatrix};$$

So

$$L = M_1^{-1} M_2^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 1.5 & 1 & 0 \\ 1.5 & 1 & 1 \end{pmatrix},$$

and

$$U = \begin{pmatrix} 2 & -1 & 1 \\ 0 & 4.5 & 7.5 \\ 0 & 0 & -4 \end{pmatrix}.$$

5c.

$$A = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 1 & 1.5 & 0 & 0 \\ 0 & -3 & 0.5 & 0 \\ 2 & -2 & 1 & 1 \end{pmatrix}.$$

$$M_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -0.5 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}, \qquad M_1 A = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 1.5 & 0 & 0 \\ 0 & -3 & 0.5 & 0 \\ 0 & -2 & 1 & 1 \end{pmatrix};$$

$$M_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 4/3 & 0 & 1 \end{pmatrix}, \qquad M_2 M_1 A = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 1.5 & 0 & 0 \\ 0 & 0 & 0.5 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix};$$

$$M_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -2 & 1 \end{pmatrix}, \qquad M_3 M_2 M_1 A = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 1.5 & 0 & 0 \\ 0 & 0 & 0.5 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix};$$

So

$$L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0.5 & 1 & 0 & 0 \\ 0 & -2 & 1 & 0 \\ 1 & -4/3 & 2 & 1 \end{pmatrix},$$

and

$$U = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 1.5 & 0 & 0 \\ 0 & 0 & 0.5 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

7a.

$$A = \begin{pmatrix} 2 & -1 & 1 \\ 3 & 3 & 9 \\ 3 & 3 & 5 \end{pmatrix}, \qquad b = \begin{pmatrix} -1 \\ 0 \\ 4 \end{pmatrix}$$

From 5*a*, *we know* $A = LU$, *where*

$$L = \begin{pmatrix} 1 & 0 & 0 \\ 1.5 & 1 & 0 \\ 1.5 & 1 & 1 \end{pmatrix},$$

and

$$U = \begin{pmatrix} 2 & -1 & 1 \\ 0 & 4.5 & 7.5 \\ 0 & 0 & -4 \end{pmatrix}$$

So

$$L\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = b.$$

we have

$y_1 = -1, \quad y_2 = -1.5, \quad y_3 = 4,$ *then*

$$U\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

$x_1 = 1, \quad x_2 = 2, \; x_3 = -1.$

7c.

$$A = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 1 & 1.5 & 0 & 0 \\ 0 & -3 & 0.5 & 0 \\ 2 & -2 & 1 & 1 \end{pmatrix}, \qquad b = \begin{pmatrix} 3 \\ 4.5 \\ -6.6 \\ 0.8 \end{pmatrix}$$

From $5a$, *we* *know* $A = LU$, *where*

$$L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0.5 & 1 & 0 & 0 \\ 0 & -2 & 1 & 0 \\ 1 & -4/3 & 2 & 1 \end{pmatrix},$$

and

$$U = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 1.5 & 0 & 0 \\ 0 & 0 & 0.5 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

So

$$L\begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} = b,$$

we *have*

$y_1 = 3,\quad y_2 = 3,\quad y_3 = -0.6,\quad y_4 = 3.$ *then*

$$U\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix}$$

$x_1 = 1.5,\quad x_2 = 2,\quad x_3 = -1.2,\quad x_4 = 3.$

6.6

3b.

$$A=\begin{pmatrix}4 & 1 & 1 & 1\\ 1 & 3 & -1 & 1\\ 1 & -1 & 2 & 0\\ 1 & 1 & 0 & 2\end{pmatrix}.$$

$$M_1=\begin{pmatrix}1 & 0 & 0 & 0\\ -1/4 & 1 & 0 & 0\\ -1/4 & 0 & 1 & 0\\ -1/4 & 0 & 0 & 1\end{pmatrix}, \qquad M_1A=\begin{pmatrix}4 & 1 & 1 & 1\\ 0 & 11/4 & -5/4 & 3/4\\ 0 & -5/4 & 7/4 & -1/4\\ 0 & 3/4 & -1/4 & 7/4\end{pmatrix};$$

$$M_2=\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 5/11 & 1 & 0\\ 0 & -3/11 & 0 & 1\end{pmatrix}, \qquad M_2M_1A=\begin{pmatrix}4 & 1 & 1 & 1\\ 0 & 11/4 & -5/4 & 3/4\\ 0 & 0 & 13/11 & 1/11\\ 0 & 0 & 1/11 & 17/11\end{pmatrix};$$

$$M_3=\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & -1/13 & 1\end{pmatrix}, \qquad M_3M_2M_1A=\begin{pmatrix}4 & 1 & 1 & 1\\ 0 & 11/4 & -5/4 & 3/4\\ 0 & 0 & 13/11 & 1/11\\ 0 & 0 & 0 & 20/13\end{pmatrix};$$

So

$$L=M_1^{-1}M_2^{-1}M_3^{-1}=\begin{pmatrix}1 & 0 & 0 & 0\\ 1/4 & 1 & 0 & 0\\ 1/4 & -5/11 & 1 & 0\\ 1/4 & 3/11 & 1/13 & 1\end{pmatrix},$$

and

$$D=\begin{pmatrix}4 & 0 & 0 & 0\\ 0 & 11/4 & 0 & 0\\ 0 & 0 & 13/11 & 0\\ 0 & 0 & 0 & 20/13\end{pmatrix}.$$

5b.

$$A = \begin{pmatrix} 4 & 1 & 1 & 1 \\ 1 & 3 & -1 & 1 \\ 1 & -1 & 2 & 0 \\ 1 & 1 & 0 & 2 \end{pmatrix}$$

From 3a, we know

$$A = BDB^t$$

where

$$B = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1/4 & 1 & 0 & 0 \\ 1/4 & -5/11 & 1 & 0 \\ 1/4 & 3/11 & 1/13 & 1 \end{pmatrix},$$

and

$$D = \begin{pmatrix} 4 & 0 & 0 & 0 \\ 0 & 11/4 & 0 & 0 \\ 0 & 0 & 13/11 & 0 \\ 0 & 0 & 0 & 20/13 \end{pmatrix}.$$

So

Let

$$L = B \cdot diag\{\sqrt{d_1}, \sqrt{d_2}, \sqrt{d_3}, \sqrt{d_4}\} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1/4 & 1 & 0 & 0 \\ 1/4 & -5/11 & 1 & 0 \\ 1/4 & 3/11 & 1/13 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & \sqrt{11/4} & 0 & 0 \\ 0 & 0 & \sqrt{13/11} & 0 \\ 0 & 0 & 0 & \sqrt{20/13} \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 0 & 0 & 0 \\ 1/2 & \sqrt{11/4} & 0 & 0 \\ 1/2 & -5/\sqrt{44} & \sqrt{13/11} & 0 \\ 1/2 & 3/\sqrt{44} & 1/\sqrt{143} & \sqrt{20/13} \end{pmatrix},$$

then

$$A = LL^t.$$

7b.

$$A = \begin{pmatrix} 4 & 1 & 1 & 1 \\ 1 & 3 & -1 & 1 \\ 1 & -1 & 2 & 0 \\ 1 & 1 & 0 & 2 \end{pmatrix}, \qquad b = \begin{pmatrix} 0.65 \\ 0.05 \\ 0 \\ 0.5 \end{pmatrix}.$$

From 3a, we know

$$A = LDL^t$$

where

$$L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1/4 & 1 & 0 & 0 \\ 1/4 & -5/11 & 1 & 0 \\ 1/4 & 3/11 & 1/13 & 1 \end{pmatrix},$$

and

$$D = \begin{pmatrix} 4 & 0 & 0 & 0 \\ 0 & 11/4 & 0 & 0 \\ 0 & 0 & 13/11 & 0 \\ 0 & 0 & 0 & 20/13 \end{pmatrix}.$$

$Ly = b$, *then we have*

$$y = \begin{pmatrix} 0.65 \\ -0.1125 \\ -0.21363636 \\ 0.38461538 \end{pmatrix}.$$

$Dz = y$, *then*

$$z = \begin{pmatrix} 0.1625 \\ -0.04090979 \\ -0.18076923 \\ 0.25 \end{pmatrix}.$$

$L^t x = z$, *then*

$$x = \begin{pmatrix} 0.2 \\ -0.2 \\ -0.2 \\ 0.25 \end{pmatrix}.$$

13.
Crout method for tridiagonal linear systems

The solution is

1.00000000	1.00000000	1.00000000	1.00000000	1.00000000
1.00000000	1.00000000	1.00000000	1.00000000	1.00000000

22.
a.

$$\det(A) = \alpha - 2$$

So, when $\det(A) = 0$, *i.e.* $\alpha = 2$, the matrix A is singular.

b. A is not strictly diagonally dominant for all $\alpha \in R$,

since $|a_{ii}| = \sum_{\substack{j=1 \\ j\neq i}}^{3} |a_{ij}|, \quad for \quad i = 1,2$

c. A is symmetric for all $\alpha \in R$.

d. For all $\alpha \in R$, A is symmetric, and

$$\det(A_1) = 1 > 0,$$
$$\det(A_2) = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 > 0,$$
$$\det(A_3) = \det(A) = \alpha - 2 > 0,$$

So A is positive definite for all $\alpha > 2$.