Introduction to Partial Differential Equations

Dr. Philip Walker

Section 1.2
Derivation of the Heat Equation in One Space Dimension

- An **interval** is a connected set of real numbers containing more than one number.

- An **interval** is a connected set of real numbers containing more than one number.
- Types of intervals:

- An **interval** is a connected set of real numbers containing more than one number.
- Types of intervals:
- $[a, b] = \{x | a \leq x \leq b\}$ when $a < b$

- An **interval** is a connected set of real numbers containing more than one number.
- Types of intervals:
- $[a, b] = \{x|a \leq x \leq b\}$ when $a < b$
- $(a, b) = \{x|a < x < b\}$

- An **interval** is a connected set of real numbers containing more than one number.
- Types of intervals:
- $[a, b] = \{x | a \leq x \leq b\}$ when $a < b$
- $(a, b) = \{x | a < x < b\}$
- $[a, b) = \{x | a \leq x < b\}$

- An **interval** is a connected set of real numbers containing more than one number.
- Types of intervals:
- $[a, b] = \{x|a \leq x \leq b\}$ when $a < b$
- $(a, b) = \{x|a < x < b\}$
- $[a, b) = \{x|a \leq x < b\}$
- $(a, b] = \{x|a < x \leq b\}$

- An **interval** is a connected set of real numbers containing more than one number.
- Types of intervals:
- $[a, b] = \{x | a \leq x \leq b\}$ when $a < b$
- $(a, b) = \{x | a < x < b\}$
- $[a, b) = \{x | a \leq x < b\}$
- $(a, b] = \{x | a < x \leq b\}$
- $[a, \infty) = \{x | a \leq x\}$

- An **interval** is a connected set of real numbers containing more than one number.
- Types of intervals:
- $[a, b] = \{x|a \leq x \leq b\}$ when $a < b$
- $(a, b) = \{x|a < x < b\}$
- $[a, b) = \{x|a \leq x < b\}$
- $(a, b] = \{x|a < x \leq b\}$
- $[a, \infty) = \{x|a \leq x\}$
- $(a, \infty) = \{x|a < x\}$

- An **interval** is a connected set of real numbers containing more than one number.
- Types of intervals:
- $[a, b] = \{x | a \leq x \leq b\}$ when $a < b$
- $(a, b) = \{x | a < x < b\}$
- $[a, b) = \{x | a \leq x < b\}$
- $(a, b] = \{x | a < x \leq b\}$
- $[a, \infty) = \{x | a \leq x\}$
- $(a, \infty) = \{x | a < x\}$
- $(-\infty, a] = \{x | x \leq a\}$

- An **interval** is a connected set of real numbers containing more than one number.
- Types of intervals:
- $[a, b] = \{x | a \leq x \leq b\}$ when $a < b$
- $(a, b) = \{x | a < x < b\}$
- $[a, b) = \{x | a \leq x < b\}$
- $(a, b] = \{x | a < x \leq b\}$
- $[a, \infty) = \{x | a \leq x\}$
- $(a, \infty) = \{x | a < x\}$
- $(-\infty, a] = \{x | x \leq a\}$
- $(-\infty, a) = \{x | x < a\}$

- An **interval** is a connected set of real numbers containing more than one number.
- Types of intervals:
- $[a, b] = \{x | a \leq x \leq b\}$ when $a < b$
- $(a, b) = \{x | a < x < b\}$
- $[a, b) = \{x | a \leq x < b\}$
- $(a, b] = \{x | a < x \leq b\}$
- $[a, \infty) = \{x | a \leq x\}$
- $(a, \infty) = \{x | a < x\}$
- $(-\infty, a] = \{x | x \leq a\}$
- $(-\infty, a) = \{x | x < a\}$
- $\mathbb{R} = (-\infty, \infty) =$ the set of real numbers

While $[a, a]$ can be used to denote the single number a
$[a, a]$ is not an interval

Theorem

If f is continuous and F is an anti-derivative of f on $[a, b]$, then

$$\int_a^b f(x)dx = F(b) - F(a).$$

Theorem

If f has a continuous derivative on $[a, b]$, then

$$\int_a^b f'(x)dx = f(b) - f(a).$$

Corollary

Suppose that f is a real valued function whose domain is a subset of the plane. If $\frac{\partial f}{\partial x}$ is continuous on the segment joining (a, c) to (b, c), then

$$\int_a^b \frac{\partial f}{\partial x}(x, c)dx = f(b, c) - f(a, c).$$

The integral is order preserving.

Theorem

If each of f and g is integrable on $[a,b]$ and $f(x) \leq g(x)$ for all x in $[a,b]$, then

$$\int_a^b f(x)dx \leq \int_a^b g(x)dx.$$

The following theorem will be used in the derivation of the heat equation one space dimension.

Theorem

Suppose that f is a continuous real valued function whose domain is an interval J. If

$$\int_a^b f(x)dx = 0$$

for every pair of numbers a and b in J with $a \leq b$, then

$$f(x) = 0 \text{ for all } x \text{ in } J.$$

Proof. We will prove the contrapositive: If $f(x_0) \neq 0$ for some x_0 in J, then $\int_a^b f(x)dx \neq 0$ for some pair of numbers a and b in J with $a \leq b$.

So, suppose that $f(x_0) \neq 0$.Then either $f(x_0) > 0$ or $f(x_0) < 0$.

If $f(x_0) > 0$, let

$$\epsilon = \frac{f(x_0)}{2}.$$

Note that $\epsilon > 0$.

Using the usual definition of continuity, let δ be a positive number such that if x is in J and $x_0 - \delta < x < x_0 + \delta$ then $f(x_0) - \epsilon < f(x) < f(x_0) + \epsilon$.

Since J is an interval, there will be numbers a and b in J with $x_0 - \delta < a < b < x_0 + \delta$.

Then when $a \leq x \leq b$ we have $f(x_0) - \epsilon < f(x) < f(x_0) + \epsilon$. Since $f(x_0) - \epsilon = f(x_0) - \frac{f(x_0)}{2} = \frac{f(x_0)}{2}$we have $\frac{f(x_0)}{2} < f(x)$ for all x in $[a, b]$. Thus $\int_a^b \frac{f(x_0)}{2} dx \leq \int_a^b f(x) dx$. Since $\int_a^b \frac{f(x_0)}{2} dx = \frac{f(x_0)}{2}(b - a) > 0$ it follows that $0 < \int_a^b f(x) dx$. Of course, this implies that $\int_a^b f(x) dx \neq 0$.

The argument for the case where $f(x_0) < 0$ is similar.

End of Proof.

A rod of length L (units of length), insulated except perhaps at its ends, lies along the x-axis with its left end at coordinate 0 and its right end at coordinate L.

Suppose that the mass density ρ (units of mass divided by units of length) and thermal conductivity K_0 ((energy$\times$length)/(time $\times$temperature)) and specific heat c (energy/(mass$\times$temperature)) at each point in the rod depend only on the x-coordinate of the point.

Let e, ϕ, and Q be as follows.

The thermal energy density (energy/length) at t (units of time after the time origin) at points with first coordinate x is $e(x, t)$.

The heat flux (energy/time) to the right at time t through the cross section consisting of points with first coordinate x is $\phi(x, t)$. (A negative value for $\phi(x, t)$ indicates heat flow to the left.)

The heat energy per unit length being generated per unit time inside the rod at time t at points with first coordinate x is $Q(x, t)$. (A negative value for Q indicates a heat sink.)

Suppose that $0 \le a \le b \le L$. Conservation of thermal energy tells us that the time-rate-of-change in thermal energy in the section of the rod consisting of points with first coordinate x satisfying $a \le x \le b$ is the net heat energy flowing per unit time across the boundaries of this section plus the net heat energy being generated internally in the section. Thus

$$\frac{d}{dt}\int_a^b e(x,t)dx = \phi(a,t) - \phi(b,t) + \int_a^b Q(x,t)dx.$$

$$\frac{d}{dt}\int_a^b e(x,t)dx = \phi(a,t) - \phi(b,t) + \int_a^b Q(x,t)dx.$$

Assuming that e and ϕ have continuous first order partial derivatives, we have

$$\int_a^b \frac{\partial e}{\partial t}(x,t)dx = -\int_a^b \frac{\partial \phi}{\partial x}(x,t)dx + \int_a^b Q(x,t)dx.$$

Thus

$$\int_a^b \left(\frac{\partial e}{\partial t}(x,t) + \frac{\partial \phi}{\partial x}(x,t) - Q(x,t)\right) dx = 0.$$

Since this is true for each choice of a and b with $0 \leq a \leq b \leq L$, if Q is continuous and e and ϕ have continuous first order partials, it follows that

$$\frac{\partial e}{\partial t}(x,t) = -\frac{\partial \phi}{\partial x}(x,t) + Q(x,t) \text{ for } 0 \leq x \leq L \text{ and } t \geq 0.$$

By definition, the temperature u and thermal energy density u are related by

$$e(x,t) = c(x)\rho(x)(u(x,t) - Z) \text{ for } 0 \leq x \leq L \text{ and } t \geq 0.$$

where Z is absolute zero on the temperature scale being used. So from

$$\frac{\partial e}{\partial t}(x,t) = -\frac{\partial \phi}{\partial x}(x,t) + Q(x,t) \text{ for } 0 \leq x \leq L \text{ and } t \geq 0.$$

we have

$$c(x)\rho(x)\frac{\partial u}{\partial t}(x,t) = -\frac{\partial \phi}{\partial x}(x,t) + Q(x,t) \text{ for } 0 \leq x \leq L \text{ and } t \geq 0.$$

According to Fourier's law of heat conduction

$$\varphi(x,t) = -K_0(x)\frac{\partial u}{\partial x}(x,t) \text{ for } 0 \leq x \leq L \text{ and } t \geq 0.$$

So from

$$c(x)\rho(x)\frac{\partial u}{\partial t}(x,t) = -\frac{\partial \phi}{\partial x}(x,t) + Q(x,t) \text{ for } 0 \leq x \leq L \text{ and } t \geq 0.$$

we arrive at **the heat diffusion equation in one space dimension**:

$$c\rho\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(K_0\frac{\partial u}{\partial x}\right) + Q \text{ for } 0 \leq x \leq L \text{ and } t \geq 0. \qquad (*)$$

$$c\rho\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(K_0\frac{\partial u}{\partial x}\right) + Q \text{ for } 0 \le x \le L \text{ and } t \ge 0. \qquad (**)$$

If each of c, ρ, and K_0 is constant and there are no internal sinks or sources so that Q is zero, we have

$$\frac{\partial u}{\partial t}(x,t) = \kappa\frac{\partial^2 u}{\partial x^2}(x,t) \text{ for } 0 \le x \le L \text{ and } t \ge 0 \qquad (*)$$

where

$$\kappa = \frac{K_0}{c\rho}$$

is the thermal diffusivity.

We will refer to (*) as **the simplified heat equation in one space dimension**.

If the rod is in thermal equilibrium

$$\frac{\partial u}{\partial t} = 0.$$

and two-place functions become one-place functions. The heat flux and temperature are related by

$$\varphi(x) = -K_0(x)u'(x).$$

Equation (**) becomes

$$0 = (K_0 u')' + Q$$

and equation (*) becomes

$$0 = u''.$$