Math 3363 Final Exam Review

Spring 2020

1. Study the problems and solutions for Homeworks 1-5 and Exams I and II.
2. Study the reviews for Exams I, II.
3. A rod of length L (units of length), insulated except perhaps at its ends, lies along the x-axis with its left end at coordinate 0 and its right end at coordinate L. Let e, ϕ, and Q be as follows. The thermal energy density (energy/length) at t (units of time after the time origin) at points with first coordinate x is $e(x,t)$. The heat flux (energy/time) to the right at time t through the cross section consisting of points with first coordinate x is $\phi(x,t)$. (A negative value for $\phi(x,t)$ indicates heat flow to the left.) The heat energy being generated per unit time inside the rod at time t at points with first coordinate x is $Q(x,t)$. (A negative value for Q indicates a heat sink.) Derive the equation

$$\frac{\partial e}{\partial t}(x,t) = -\frac{\partial \phi}{\partial x}(x,t) + Q(x,t) \text{ for } 0 \le x \le L \text{ and } t \ge 0.$$

   Let $u(x,t)$ be the temperature at points with first coordinate x at time t, $c(x)$ be the specific heat, $\rho(x)$ be the mass density, and $K_0(x)$ be the thermal conductivity at points with first coordinate x. Derive the equation

$$c\rho\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}(K_0\frac{\partial u}{\partial x}) + Q \text{ for } 0 \le x \le L \text{ and } t \ge 0.$$

4. Derive the corresponding equations in two space dimensions.
5. Find the function u such that $u''(x) = 1 + x$ for $0 \le x \le 2$, $u(0) = -1$, and $u(2) = 4$.
6. Find the value of β for which the following problem has an equilibrium temperature distribution.

$$\frac{\partial w}{\partial t}(x,t) = \frac{\partial^2 w}{\partial x^2}(x,t) + x \text{ for } t \ge 0 \text{ and } 0 \le x \le L,$$

$$w(x,0) = f(x) \text{ for } 0 \le x \le L,$$

$$\frac{\partial w}{\partial x}(0,t) = 1, \text{ and } \frac{\partial w}{\partial x}(L,t) = \beta \text{ for } t \ge 0.$$

Let u be the equilibrium solution so that

$$u(x) = \lim_{t \to \infty} w(x,t) \text{ for } 0 \le x \le L.$$

Find a formula for $u(x)$ that does not contain any undetermined constants.

7. Consider the following two-point boundary value problem in which L is a positive number.

$$\begin{array}{ll} \text{(i)} & -\varphi''(x) = \lambda\varphi(x) \quad \text{for } 0 \le x \le L, \\ \text{(ii)} & \varphi'(0) = 0, \text{ and} \\ \text{(iii)} & \varphi(L) = 0. \end{array}$$

Use the Rayleigh Quotient to show that all eigenvalues are non negative. How do you know that 0 is not an eigenvalue?

8. Find 2×2 matrices M and N so that conditions (i) and (ii) given in the previous problem are equivalent to

$$M \begin{bmatrix} \varphi(0) \\ \varphi'(0) \end{bmatrix} + N \begin{bmatrix} \varphi(L) \\ \varphi'(L) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

9. For the two-point boundary value problem given in problem 7, find the matrix $D(\lambda)$ and the determinant $\Delta(\lambda)$ in the case where $\lambda > 0$.

10. For the two-point boundary value problem given in Problem 7, find a proper listing of eigenvalues and eigenfunctions.

11. Know how to solve second order linear homogeneous constant coefficient differential equations. i.e. equations of the form

$$ay''(x) + by'(x) + cy(x) = 0$$

for all x in an interval J.

12. Know how to solve the Cauchy-Euler differential equation:

$$ax^2y''(x) + bxy'(x) + cy(x) = 0$$

for all $x > 0$.

13. Suppose that $\{\phi_k\}_{k=1}^n$ is orthogonal on $[a,b]$ and $< \phi_k, \phi_k > \neq 0$ for $k = 1, \ldots, n$. Suppose that $f = \sum_{k=1}^n c_k\phi_k$. Derive a formula that gives c_k in terms of f, ϕ_k, and the inner product.

14. Derive the solution to

$$\begin{aligned} \frac{\partial u}{\partial t}(x,t) &= \kappa\frac{\partial^2 u}{\partial x^2}(x,t) \text{ for } t \ge 0 \text{ and } a \le x \le b, \\ u(a,t) &= 0 \text{ for } t \ge 0, \\ u(b,t) &= 0 \text{ for } t \ge 0, \text{ and} \\ u(x,0) &= f(x) \text{ for } a \le x \le b \end{aligned}$$

where $a < b$ and κ is a positive number.

15. Derive the solution to

$$\begin{aligned}
\frac{\partial u}{\partial t}(x,t) &= \kappa\frac{\partial^2 u}{\partial x^2}(x,t) \text{ for } t \geq 0 \text{ and } 0 \leq x \leq L, \\
\frac{\partial u}{\partial x}(0,t) &= 0 \text{ for } t \geq 0, \\
u(L,t) &= 0 \text{ for } t \geq 0, \text{ and} \\
u(x,0) &= f(x) \text{ for } 0 \leq x \leq L
\end{aligned}$$

where each of κ and L is a positive number.

16. Find the solution to

$$\begin{aligned}
\frac{\partial u}{\partial t}(x,t) &= \kappa\frac{\partial^2 u}{\partial x^2}(x,t) \text{ for } t \geq 0 \text{ and } 0 \leq x \leq 1, \\
u(0,t) &= u(1,t) = 0 \text{ for } t \geq 0, \text{ and} \\
u(x,0) &= \sin \pi x \text{ for } 0 \leq x \leq 1.
\end{aligned}$$

17. Sketch the graphs where $y = \dfrac{x}{2}$ with $x > 0$ and where $y = \tan x$ with $x > 0$ on the same set of axes and explain how to find numerical approximations to the first two numbers x such that

$$2\sin x - x\cos x = 0.$$

18. Suppose that each of L and H is a positive number. Derive the solution to

$$\begin{aligned}
\frac{\partial^2 u}{\partial x^2}(x,y) + \frac{\partial^2 u}{\partial y^2}(x,y) &= 0 \text{ for } 0 \leq x \leq L \text{ and } 0 \leq y \leq H, \\
\frac{\partial u}{\partial x}(0,y) &= \frac{\partial u}{\partial x}(L,y) = 0 \text{ for } 0 \leq y \leq H, \\
\frac{\partial u}{\partial y}(x,H) &= 0, \text{ and } u(x,0) = f(x) \text{ for } 0 \leq x \leq L.
\end{aligned}$$

19. Suppose that each of c and L is a positive number. Derive the solution to

$$\begin{aligned}
\frac{\partial^2 u}{\partial t^2}(x,t) &= c^2\frac{\partial^2 u}{\partial x^2}(x,t) \text{ for } 0 \leq x \leq L \text{ and all } t \text{ in } \mathbb{R}, \\
u(0,t) &= 0 \text{ for all } t \text{ in } \mathbb{R}, \\
u(L,t) &= 0 \text{ for all } t \text{ in } \mathbb{R}, \\
u(x,0) &= f(x) \text{ for } 0 \leq x \leq L, \text{ and} \\
\frac{\partial u}{\partial t}(x,0) &= g(x) \text{ for } 0 \leq x \leq L.
\end{aligned}$$

20. Let

$$f(x) = \begin{cases} 1 - x & \text{when} \quad -1 < x < 0 \\ x & \text{when} \quad 0 < x < 1 \end{cases}.$$

Find the Fourier series for f.

21. Let $\{S_n\}$ be the Fourier series for the function f in the previous problem and let

$$g(x) = \lim_{n\to\infty} S_n(x) \text{ for } -5 \le x \le 5.$$

Sketch the graph of g. Be sure to indicate the value of g at the numbers where g is discontinuous.

22. Suppose that L is a positive number.

  (a) Define the Fourier series for f when f is defined on $[-L, L]$.

  (b) Define the cosine series for f when f is defined on $[0, L]$.

  (c) Define the sine series for f when f is defined on $[0, L]$.

23. Let $f(x) = 1 - x^2$ for $0 < x < 1$.

  (a) Sketch the function to which the cosine series of f converges on $[-4, 4]$.

  (b) Sketch the function to which the sine series of f converges on $[-4, 4]$.

24. Let

$$f(x) = \begin{cases} -1 & \text{when} \quad -1 < x < 0 \\ 1 & \text{when} \quad 0 < x < 1 \end{cases}, \ .$$

and let $\{S_n\}_{n=1}^{\infty}$ be the trigonometric Fourier series for f. Sketch the graph of f and the graph of a typical S_n on the same set of axes. Describe the Gibbs phenomenon.

25. Let f and $\{S_n\}_{n=1}^{\infty}$ be as in the previous problem. Explain why $\{S_n\}_{n=1}^{\infty}$ does not converge uniformly.

26. State the Parseval identities for the sine series, the cosine series, and the Fourier series.

27. State the Weierstrass M-test.

28. Suppose that γ is a negative number. Find

$$\sum_{k=1}^{\infty} e^{\gamma k}.$$

Suggestion:

$$e^{\gamma k} = \left(e^{\gamma}\right)^k .$$

29. Let D be the set of all (x, t) such that x is a real number and $t \ge 1$; let $\{\alpha_k\}_{k=1}^{\infty}$ be a sequence of real numbers; let $\{\beta_k\}_{k=1}^{\infty}$ be a sequence of real numbers with $\beta_k \ge k$ for $k = 1, 2, \ldots$; and let let $\{E_k\}$ be a bounded sequence of real numbers. Let

$$U_n(x, t) = \sum_{k=1}^{n} E_k e^{-\beta_k t} \sin a_k x$$

for all (x, t) in D and $n = 1, 2, \ldots$. Show that $\{U_n\}_{n=1}^{\infty}$ converges uniformly on D.

30. Suppose that each of L and λ is a positive number. Find the function G that satisfies

$$\begin{aligned} G''(x) &= \lambda G(x) \text{ for } 0 \le x \le L, \\ G(0) &= 0, \text{ and} \\ G(L) &= 1. \end{aligned}$$

This is not an eigenvalue problem.

31. Suppose that each of L and H is a positive number. Derive the solution to

$$\begin{aligned} \frac{\partial^2 u}{\partial x^2}(x, y) + \frac{\partial^2 u}{\partial y^2}(x, y) &= 0 \text{ for } 0 \le x \le L \text{ and } 0 \le y \le H, \\ u(0, y) &= u(L, y) = 0 \text{ for } 0 \le y \le H, \\ u(x, 0) &= f(x), \text{ and } u(x, H) = 0 \text{ for } 0 \le x \le L. \end{aligned}$$

32. Suppose that each of L and H is a positive number. Derive the solution to

$$\begin{aligned} \frac{\partial^2 u}{\partial x^2}(x, y) + \frac{\partial^2 u}{\partial y^2}(x, y) &= 0 \text{ for } 0 \le x \le L \text{ and } 0 \le y \le H, \\ u(0, y) &= u(L, y) = 0 \text{ for } 0 \le y \le H, \\ u(x, H) &= f(x), \text{ and } u(x, 0) = 0 \text{ for } 0 \le x \le L. \end{aligned}$$

33. Suppose that each of L and H is a positive number. Derive the solution to

$$\begin{aligned} \frac{\partial^2 u}{\partial x^2}(x, y) + \frac{\partial^2 u}{\partial y^2}(x, y) &= 0 \text{ for } 0 \le x \le L \text{ and } 0 \le y \le H, \\ u(0, y) &= f(y) \text{ and } u(L, y) = 0 \text{ for } 0 \le y \le H \\ u(x, H) &= 0 \text{ and } u(x, 0) = 0 \text{ for } 0 \le x \le L. \end{aligned}$$

34. Suppose that each of L and H is a positive number. Derive the solution to

$$\begin{aligned} \frac{\partial^2 u}{\partial x^2}(x, y) + \frac{\partial^2 u}{\partial y^2}(x, y) &= 0 \text{ for } 0 \le x \le L \text{ and } 0 \le y \le H, \\ u(0, y) &= 0 \text{ and } u(L, y) = f(y) \text{ for } 0 \le y \le H, \\ u(x, H) &= 0 \text{ and } u(x, 0) = 0 \text{ for } 0 \le x \le L. \end{aligned}$$

35. Find the function v of the form

$$v(x, y) = ax + by + cxy + d$$

such that

$$\begin{aligned}
v(0,0) &= -1, \\
v(2,0) &= 3, \\
v(2,4) &= 4, \text{ and} \\
v(0,4) &= -2.
\end{aligned}$$

36. Suppose that each of L and H is a positive number. Derive the solution to

$$\begin{aligned}
\frac{\partial^2 u}{\partial x^2}(x,y) + \frac{\partial^2 u}{\partial y^2}(x,y) &= 0 \text{ for } 0 \le x \le 2 \text{ and } 0 \le y \le 1, \\
u(0,y) &= y^2 + y + 4 \text{ and } u(2,y) = 8y + 6 \text{ for } 0 \le y \le 1, \\
u(x,H) &= 4x + 6 \text{ and } u(x,0) = x^2 - x + 4 \text{ for } 0 \le x \le 2.
\end{aligned}$$

In order to improve the convergence of the series solution, do this by first finding a function v of the form

$$v(x,y) = ax + by + cxy + d$$

that agrees with the given boundary conditions at the four corners of the rectangle. Then let

$$w(x,y) = u(x,y) - v(x,y)$$

for all (x,y)in the rectangle. Calculate the boundary conditions for w (w will be zero at the four corners) and noting that w is also a solution to Laplace's equation find the function w. Find u by noting that $u = w + v$.

37. Suppose that L is a positive number and that each of k and j is an integer with $k \ge 0$ and $j > 0$. Evaluate

$$\int_{-L}^{L} \cos\frac{k\pi x}{L} \sin\frac{j\pi x}{L} dx.$$

38. Let

$$f(x) = \begin{cases} x & \text{when} \quad -1 < x < 0 \\ 2 - x & \text{when} \quad 0 < x < 1 \end{cases}.$$

Find the Fourier series for f, the sine series for f, and the cosine series for f. In each case take $L = 1$.

39. Do the following problems from the text.

    (a) 2.5.1 and 2.5.2 pages 81 and 82.
    (b) 4.4.1 page 140
    (c) 3.2.1 and 3.2.2 page 92

(d) 3.3.1 page 110

40. Derive the solution to

$$\begin{aligned}
\frac{\partial u}{\partial t}(x,t) &= \kappa \frac{\partial^2 u}{\partial x^2}(x,t) + Q(x,t) \text{ for } t \geq 0 \text{ and } 0 \leq x \leq L, \\
a\frac{\partial u}{\partial x}(0,t) + bu(0,t) &= 0 \text{ for } t \geq 0, \\
c\frac{\partial u}{\partial x}(L,t) + du(L,t) &= 0 \text{ for } t \geq 0, \\
u(x,0) &= f(x) \text{ for } 0 \leq x \leq L.
\end{aligned}$$

41. Find the solution to

$$\begin{aligned}
\frac{\partial u}{\partial t}(x,t) &= \frac{\partial^2 u}{\partial x^2}(x,t) + e^{-t}\sin \pi x \text{ for } t \geq 0 \text{ and } 0 \leq x \leq 1, \\
u(0,t) &= 0 \text{ for } t \geq 0, \\
u(1,t) &= 0 \text{ for } t \geq 0, \\
u(x,0) &= \sin \pi x \text{ for } 0 \leq x \leq 1.
\end{aligned}$$

42. Derive d'Alembert's solution to the wave equation.

43. Let u be the solution to

$$\begin{aligned}
\frac{\partial^2 u}{\partial t^2}(x,t) &= \frac{\partial^2 u}{\partial x^2}(x,t) \text{ for all } x \text{ and } t \text{ in } \mathbb{R}, \\
u(x,0) &= \varphi(x) \text{ for all } x \text{ in } \mathbb{R}, \text{ and} \\
\frac{\partial u}{\partial t}(x,0) &= 0 \text{ for all } x \text{ in } \mathbb{R}.
\end{aligned}$$

where

$$\varphi(x) = \left\{ \begin{array}{ccc} 0 & \text{for} & x < 0 \\ 2x & \text{for} & 0 \leq x \leq 1 \\ 4-2x & \text{for} & 1 \leq x \leq 2 \\ 0 & \text{for} & x > 2 \end{array} \right. .$$

Let

$$h(x) = u(x,3) \text{ for all } x \text{ in } \mathbb{R}.$$

Sketch the graph of h.

44. Let u be the solution to

$$\begin{aligned}
\frac{\partial^2 u}{\partial t^2}(x,t) &= \frac{\partial^2 u}{\partial x^2}(x,t) \text{ for all } x \text{ and } t \text{ in } \mathbb{R}, \\
u(x,0) &= 0 \text{ for all } x \text{ in } \mathbb{R}, \text{ and} \\
\frac{\partial u}{\partial t}(x,0) &= \psi(x) \text{ for all } x \text{ in } \mathbb{R}.
\end{aligned}$$

where

$$\psi(x) = \begin{cases} 0 & \text{for} \quad x < 0 \\ 2 & \text{for} \quad 0 < x < 1 \\ -2 & \text{for} \quad 1 < x < 2 \\ 0 & \text{for} \quad x > 2 \end{cases}.$$

Let

$$h(x) = u(x, 3) \text{ for all } x \text{ in } \mathbb{R}.$$

Sketch the graph of h.

45. Do problems 7.3.1(c), 7.3.4(b), and 7.3.7(c) on pages 278-281 of the text.

46. Derive the solution to Laplace's Equation in polar coordinates in an annulus.

47. Derive the eigenvalues and eigenfunctions for the two-dimensional rectangular problem

$$-\nabla^2\varphi = \lambda\varphi \text{ on } [0, L] \times [0, H]$$

with various boundary conditions such as

$$\varphi = 0 \text{ on the boundary of } [0, L] \times [0, H].$$

48. Derive the solution to the heat equation and the wave equation for a rectangle with various boundary conditions.