Math 3363 Final Examination Solutions

Spring 2020

Do the problems in the order in which they are listed. Upload your solutions in a pdf file by 11:59 p.m. Thursday, April 30. You may use your text, notes, and the material posted on Dr. Walker's web site, but you must DO YOUR OWN WORK.

You may use the following information without derivation.

- A proper listing of eigenvalues and eigenfunctions for

$$\begin{array}{ll} \text{(i)} & -\varphi''(x) = \lambda\varphi(x) \quad \text{for } 0 \le x \le L, \\ \text{(ii)} & \varphi(0) = 0, \text{ and} \\ \text{(iii)} & \varphi(L) = 0 \end{array}$$

  is $\{\lambda_k\}_{k=1}^{\infty}$ and $\{\varphi_k\}_{k=1}^{\infty}$ where $\lambda_k = (\frac{k\pi}{L})^2$ and $\varphi_k(x) = \sin\frac{k\pi x}{L}$.

- A proper listing of eigenvalues and eigenfunctions for

$$\begin{array}{ll} \text{(i)} & -\varphi''(x) = \lambda\varphi(x) \quad \text{for } 0 \le x \le L, \\ \text{(ii)} & \varphi'(0) = 0, \text{ and} \\ \text{(iii)} & \varphi'(L) = 0 \end{array}$$

  is $\{\lambda_k\}_{k=0}^{\infty}$ and $\{\varphi_k\}_{k=0}^{\infty}$ where $\lambda_k = (\frac{k\pi}{L})^2$ and $\varphi_k(x) = \cos\frac{k\pi x}{L}$. Note that $\lambda_0 = 0$ and $\varphi_0(x) = 1$.

- A proper listing of eigenvalues and eigenfunctions for

$$\begin{array}{ll} \text{(i)} & -\varphi''(x) = \lambda\varphi(x) \quad \text{for } 0 \le x \le L, \\ \text{(ii)} & \varphi(0) = 0, \text{ and} \\ \text{(iii)} & \varphi'(L) = 0 \end{array}$$

  is $\{\lambda_k\}_{k=1}^{\infty}$ and $\{\varphi_k\}_{k=1}^{\infty}$ where $\lambda_k = \left(\frac{(2k-1)\pi}{2L}\right)^2$ and $\varphi_k(x) = \sin\frac{(2k-1)\pi x}{2L}$.

- A proper listing of eigenvalues and eigenfunctions for

$$\begin{array}{ll} \text{(i)} & -\varphi''(x) = \lambda\varphi(x) \quad \text{for } 0 \le x \le L, \\ \text{(ii)} & \varphi'(0) = 0, \text{ and} \\ \text{(iii)} & \varphi(L) = 0 \end{array}$$

  is $\{\lambda_k\}_{k=1}^{\infty}$ and $\{\varphi_k\}_{k=1}^{\infty}$ where $\lambda_k = \left(\frac{(2k-1)\pi}{2L}\right)^2$ and $\varphi_k(x) = \cos\frac{(2k-1)\pi x}{2L}$.

- A proper listing of eigenvalues and eigenfunctions for

$$\begin{array}{lll} \text{(i)} & -\nabla^2\varphi(x,y) = \lambda\varphi(x,y) & \text{for } 0 \le x \le L \text{ and } 0 \le y \le H \text{ and} \\ \text{(ii)} & \varphi(x,y) = 0 & \text{for } (x,y) \text{ on the boundary of } [0,L] \times [0,H] \end{array}$$

is $\{\lambda_{kj}\}_{k,j=1}^{\infty}$ and $\{\varphi_{kj}\}_{k,j=1}^{\infty}$ where $\lambda_{kj} = (\frac{k\pi}{L})^2 + (\frac{j\pi}{H})^2$ and $\varphi_{kj}(x,y) = \sin\frac{k\pi x}{L}\sin\frac{j\pi y}{H}$

1. Consider the following nonhomogeneous time dependent heat equation problem.

$$\frac{\partial u}{\partial t}(x,t) = \frac{\partial^2 u}{\partial x^2}(x,t) + \beta \text{ for } t \ge 0 \text{ and } 0 \le x \le 1$$

$$u(x,0) = 1 \text{ for } 0 \le x \le 1,$$

$$\frac{\partial u}{\partial x}(0,t) = -1, \text{ and } \frac{\partial u}{\partial x}(1,t) = 1 \text{ for } t \ge 0.$$

(a) Find the constant β so that the problem has an equilibrium solution v.

Solution.

$$\begin{aligned} 0 &= v'' + \beta, v'(0) = -1 \text{ and } v'(1) = 1 \\ \int_0^1 v''(x)dx &= \int_0^1 (-\beta)dx \\ v'(1) - v'(0) &= -\beta \\ 1 - (-1) &= -\beta \\ \beta &= -2 \end{aligned}$$

(b) Find the equilibrium solution v with no undetermined constants.

Solution.

$$\begin{aligned} v'' &= 2,\ v'(0) = -1 \text{ and } v'(1) = 1 \\ v'(x) &= 2x + c_1,\ c_1 = -1 \\ v'(x) &= 2x - 1 \\ v(x) &= x^2 - x + c_2 \end{aligned}$$

$$\begin{aligned}
\frac{d}{dt}\int_0^1 u(x,t)dx &= \int_0^1 \frac{\partial u}{\partial t}(x,t)dx \\
&= \int_0^1 (\frac{\partial^2 u}{\partial x^2}(x,t) - 2)dx \\
&= \frac{\partial u}{\partial x}(1,t) - \frac{\partial u}{\partial x}(0,t) - 2 \\
&= 1 - (-1) - 2 = 0
\end{aligned}$$

$\int_0^1 u(x,t)dx$ is constant in t.

$$\begin{aligned}
1 &= \int_0^1 (1)dx = \int_0^1 u(x,0)dx = \int_0^1 u(x,t)dx \text{ (any } t) \\
&= \lim_{t\to\infty} \int_0^1 u(x,t)dx = \int_0^1 \lim_{t\to\infty} u(x,t)dx = \int_0^1 v(x)dx = \\
&= \int_0^1 (x^2 - x + c_2)dx = \frac{1}{3} - \frac{1}{2} + c_2
\end{aligned}$$

$$c_2 = \frac{7}{6}$$

$$v(x) = x^2 - x + \frac{7}{6}$$

2. Let $w(x,t) = u(x,t) - v(x)$ where u and v are as in Problem 1.

(a) State the problem for w. Use the value of β found in Problem 1.

Solution.

$$\frac{\partial w}{\partial t}(x,t) = \frac{\partial^2 w}{\partial x^2}(x,t) \text{ for } t \geq 0 \text{ and } 0 \leq x \leq 1$$

$$w(x,0) = u(x,0) - v(x) = 1 - (x^2 - x + \frac{7}{6}) \text{ for } 0 \leq x \leq 1,$$

$$w(x,0) = -x^2 + x - \frac{1}{6}$$

$$\frac{\partial w}{\partial x}(0,t) = 0, \text{ and } \frac{\partial w}{\partial x}(1,t) = 0 \text{ for } t \geq 0.$$

(b) Solve the problem for w, then give the solution u. There should be no undetermined constants in your solution u.

Solution. Using the solution in Section 2.4,

$$w(x,t) = A_0 + \sum_{k=}^{\infty} A_k \cos k\pi x e^{-(k\pi)^2 t}$$

where

$$A_0 = \frac{1}{1}\int_0^1 g(x)dx = \int_0^1 (-x^2 + x - \frac{1}{6})dx = 0$$

and

$$A_k = \frac{2}{1}\int_0^1 (-x^2 + x - \frac{1}{6}) \cos k\pi x dx = -\frac{2}{\pi^2 k^2}((-1)^k + 1)$$

$$w(x,t) = -\frac{2}{\pi^2}\sum_{k=}^{\infty} \frac{1}{k^2}((-1)^k + 1)(\cos k\pi x)e^{-(k\pi)^2 t}$$

hence

$$u(x,t) = w(x,t) + v(x) =$$

$$u(x,t) = x^2 - x + \frac{7}{6} - \frac{2}{\pi^2}\sum_{k=}^{\infty} \frac{1}{k^2}((-1)^k + 1)(\cos k\pi x)e^{-(k\pi)^2 t}$$

3. Consider the following two-point boundary value problem.

$$\begin{array}{lll} \text{(i)} & -\varphi''(x) = \lambda\varphi(x) & \text{for } 0 \le x \le 1, \\ \text{(ii)} & \varphi(0) - \varphi'(0) = 0, \text{ and} & \\ \text{(iii)} & \varphi(1) = 0. & \end{array}$$

(a) Find 2×2 matrices M and N so that conditions (ii) and (iii) are equivalent to

$$M\begin{bmatrix} \varphi(0) \\ \varphi'(0) \end{bmatrix} + N\begin{bmatrix} \varphi(1) \\ \varphi'(1) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Solution.

$$M = \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} \text{ and } N = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

(b) Use the Rayleigh Quotient to show that all eigenvalues are non-negative. How do you know that 0 is not an eigenvalue?

Solution. Suppose that λ is an eigenvalue and φ is a corresponding eigenfunction. Then

$$\begin{aligned} \lambda &= \frac{\varphi(0)\varphi'(0) - \varphi(1)\varphi'(1) + \int_0^1 (\varphi'(x))^2 dx}{\int_0^1 (\varphi(x))^2 dx} \\ &= \frac{\varphi(0)\cdot\varphi(0) - 0\cdot\varphi'(1) + \int_0^1 (\varphi'(x))^2 dx}{\int_0^1 (\varphi(x))^2 dx} \\ &= \frac{(\varphi(0))^2 + \int_0^1 (\varphi'(x))^2 dx}{\int_0^1 (\varphi(x))^2 dx} \ge 0 \end{aligned}$$

Suppose that φ is a solution to (i), (ii), and (iii) when $\lambda = 0$. From (i),

$$\varphi'(x) = c_1$$

and

$$\varphi(x) = c_1 x + c_2.$$

Then from (ii),

$$c_2 - c_1 = 0,$$

and from (iii),

$$c_1 + c_2 = 0\ ;$$

so

$$c_1 = c_2 = 0.$$

Thus

$$\varphi(x) = 0 \text{ for } 0 \le x \le 1.$$

Since the only solution is the zero function, the number zero is not an eigenvalue.

(c) Find the matrix $D(\lambda)$ and the determinant $\Delta(\lambda)$ in the case where $\lambda > 0$.

Solution.

$$D(\lambda) = \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} \Phi_\lambda(0) + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \Phi_\lambda(1)$$

where

$$\Phi_\lambda(x) = \begin{pmatrix} \cos\sqrt{\lambda}x & \sin\sqrt{\lambda}x \\ -\sqrt{\lambda}\sin\sqrt{\lambda}x & \sqrt{\lambda}\cos\sqrt{\lambda}x \end{pmatrix}.$$

$$D(\lambda) = \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & \sqrt{\lambda} \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \cos\sqrt{\lambda} & \sin\sqrt{\lambda} \\ -\sqrt{\lambda}\sin\sqrt{\lambda} & \sqrt{\lambda}\cos\sqrt{\lambda} \end{pmatrix},$$

so

$$D(\lambda) = \begin{pmatrix} 1 & -\sqrt{\lambda} \\ \cos\sqrt{\lambda} & \sin\sqrt{\lambda} \end{pmatrix}$$

and

$$\Delta(\lambda) = \det D(\lambda) = \sin\sqrt{\lambda} + \sqrt{\lambda}\cos\sqrt{\lambda}$$

(d) Explain how to determine the eigenvalues graphically. Give a proper listing of eigenvalues and eigenfunctions.

Solution. λ is an eigenvalue if and only if $\lambda = \rho^2$ where ρ is the first coordinate of a point of intersection of the graph where $y = -x$ and the graph where $y = \tan x$.

When λ_k is an eigenvalue,

$$D(\lambda_k)\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

if and only if

$$c_1 - \sqrt{\lambda_k}c_2 = 0$$

so a corresponding eigenfunction is φ_k where

$$\varphi_k(x) = \cos\sqrt{\lambda_k}x + \frac{1}{\sqrt{\lambda_k}}\cos\sqrt{\lambda_k}x$$

4. Suppose that $\{\phi_k\}_{k=1}^{\infty}$ is orthogonal on $[0, L]$ and $< \phi_k, \phi_k > \neq 0$ for $k = 1, \ldots$. Suppose that $f = \sum_{k=1}^{\infty} c_k\phi_k$ with convergence in the mean. Derive a formula that gives c_k in terms of f, ϕ_k, and the inner product.

Solution.

$$< f, \phi_k > = < \sum_{j=1}^{\infty} c_j\phi_{j,}\phi_k > .$$

Since there is convergence in the mean,

$$< f, \phi_k > = \sum_{j=1}^{\infty} c_j < \phi_j, \phi_k > .$$

Since $\{\phi_k\}_{k=1}^{\infty}$ is orthogonal,

$$< f, \phi_k > = c_k < \phi_k, \phi_k > .$$

Thus

$$c_k = \frac{< f, \phi_k >}{< \phi_k, \phi_k >}.$$

5. Suppose that each of a, b, c, and d is a real number, at least one of a and b is not zero and at least one of c and d is not zero. Suppose that each of L and κ is a positive number, Derive the solution to

$$\begin{aligned}
\frac{\partial u}{\partial t}(x,t) &= \kappa\frac{\partial^2 u}{\partial x^2}(x,t) \text{ for } t \geq 0 \text{ and } 0 \leq x \leq L, && (1)\\
au(0,t) + b\frac{\partial u}{\partial x}(0,t) &= 0 \text{ for } t \geq 0, && (2)\\
cu(L,t) + d\frac{\partial u}{\partial x}(L,t) &= 0 \text{ for } t \geq 0, \text{ and} && (3)\\
u(x,0) &= f(x) \text{ for } 0 \leq x \leq L. && (4)
\end{aligned}$$

Let $\{\lambda_k\}_{k=1}^{\infty}$ and $\{\varphi_k\}_{k=1}^{\infty}$ be a proper listing of eigenvalues and eigenfunctions for the related Sturm-Liouville problem.

Solution. Suppose that u is an elementary separated solution to (1). This means

$$u(x,t) = \varphi(x)G(t)$$

for some pair of one-place functions φ and G. Inserting this into (1), we have

$$\varphi(x)G'(t) = \kappa\varphi''(x)G(t). \tag{5}$$

Assuming for now that

$$u(x.t) \neq 0,$$

and dividing each side of (5) by $\varphi(x)G(t)$, we have

$$\frac{\varphi(x)G'(t)}{\varphi(x)G(t)} = \kappa\frac{\varphi''(x)G(t)}{\varphi(x)G(t)},$$

so

$$\frac{G'(t)}{G(t)} = \kappa\frac{\varphi''(x)}{\varphi(x)}.$$

This holds for all $t \geq 0$ and x with $0 \leq x \leq L$, so there is a constant C such that

$$\frac{G'(t)}{G(t)} = C = \kappa\frac{\varphi''(x)}{\varphi(x)} \tag{6}$$

for all $t \geq 0$ and x with $0 \leq x \leq L$. As a matter of notational convenience and so that we can more easily make use of our earlier work on two-point boundary value problems, we let

$$\lambda = -\frac{C}{\kappa}.$$

From (6) we then have

$$-\varphi''(x) = \lambda\varphi(x) \text{ for all } x \text{ in } [0,L] \tag{7}$$

and

$$G'(t) = -\kappa\lambda G(t) \text{ for all } t \geq 0. \tag{8}$$

It is worth noting that if

$$u(x,t) = \varphi(x)G(t)$$

and (7) and (8) hold, then

$$\begin{aligned}\frac{\partial u}{\partial t}(x,t) &= \varphi(x)G'(t) = -\varphi(x)\kappa\lambda G(t)\\ &= \kappa\varphi''(x)G(t) = \kappa\frac{\partial^2 u}{\partial x^2}(x,t)\end{aligned}$$

so the PDE (1)

$$\frac{\partial u}{\partial t}(x,t) = \kappa\frac{\partial^2 u}{\partial x^2}(x,t)$$

will be satisfied, and we no longer need to assume that $u(x,t) \neq 0$. Continuing with our assumption that

$$u(x,t) = \varphi(x)G(t)$$

we have from conditions (2) and (3) that

$$a\varphi(0) + b\varphi'(0) = 0 \tag{9}$$

and

$$c\varphi(L) + d\varphi'(L) = 0. \tag{10}$$

Let

$$\{\lambda_k\}_{k=1}^{\infty} \text{ and } \{\varphi_k\}_{k=1}^{\infty}$$

be a proper listing of eigenvalues and eigenfunctions for (7), (9), and (10). When

$$\lambda = \lambda_k$$

the solutions to (8) are constant multiples of G_k where

$$G_k(t) = e^{-\kappa\lambda_k t}.$$

The problem consisting of (1), (2), and (3) is linear and homogeneous, so if $\{B_k\}_{k=1}^{n}$is a finite sequence of numbers and

$$u(x,t) = \sum_{k=1}^{n} B_k\varphi_k(x)G_k(t),$$

then u will be a solution to (1), (2), and (3). Thus we hope that the solution to the problem consisting of (1) through (4) will be of the form

$$u(x,t) = \sum_{k=1}^{\infty} B_k\varphi_k(x)G_k(t)$$

for some sequence of constants $\{B_k\}_{k=1}^{\infty}$. Noting that $G_k(0) = 1$, we see that condition (4),

$$u(x,0) = f(x) \text{ for } x \text{ in } [0,L],$$

implies

$$f = \sum_{k=1}^{\infty} B_k\varphi_k.$$

Since $\{\varphi_k\}_{k=1}^{\infty}$ is an orthogonal sequence of nonzero functions, this implies

$$B_k = \frac{< f, \varphi_k >}{< \varphi_k, \varphi_k >}$$

for $k = 1, 2, \ldots$In summary, the solution to the original problem (1) through (4) is u where

$$u(x,t) = \sum_{k=1}^{\infty} B_k\varphi_k(x)e^{-\kappa(\frac{k\pi}{L})^2 t}$$

in which

$$B_k = \frac{\int_0^L f(x)\varphi_k(x)dx}{\int_0^L (\varphi_k(x))^2 dx} \text{ for } k = 1, 2, \ldots$$

6. Suppose that each of a, b, c, and d is a real number, at least one of a and b is not zero and at least one of c and d is not zero. Suppose that each of L and H is a positive number. Derive the solution to

$$
\begin{aligned}
\frac{\partial^2 u}{\partial x^2}(x,y) + \frac{\partial^2 u}{\partial y^2}(x.y) &= 0 \text{ for } 0 \leq x \leq L \text{ and } 0 \leq y \leq H, && (1)\\
au(0,y) + b\frac{\partial u}{\partial x}(0,y) &= 0 \text{ for } 0 \leq y \leq H, && (2)\\
cu(L,y) + d\frac{\partial u}{\partial x}(L,y) &= 0 \text{ for } 0 \leq y \leq H, && (3)\\
u(x,H) &= 0 \text{ for } 0 \leq x \leq L, \text{ and} && (4)\\
u(x,0) &= f(x) \text{ for } 0 \leq x \leq L. && (5)
\end{aligned}
$$

Let $\{\lambda_k\}_{k=1}^{\infty}$ and $\{\varphi_k\}_{k=1}^{\infty}$ be a proper listing of eigenvalues and eigenfunctions for the related Sturm-Liouville problem. Suppose that all of the eigenvalues are positive.

Solution. Suppose that u is an elementary separated solution to (1). This means

$$u(x,y) = \varphi(x)h(y) \tag{6}$$

for some pair of one-place functions φ and h. Inserting this into (1), we have

$$\varphi''(x)h(y) + \varphi(x)h''(y) = 0. \tag{7}$$

Assuming for now that

$$u(x.y) \neq 0,$$

and dividing each side of (7) by $\varphi(x)h(y)$, we have

$$\frac{\varphi''(x)h(y)}{\varphi(x)h(y)} + \frac{\varphi(x)h''(y)}{\varphi(x)h(y)} = 0,$$

so

$$\frac{h''(y)}{h(y)} = -\frac{\varphi''(x)}{\varphi(x)}.$$

This holds for all y with $0 \leq y \leq H$ and x with $0 \leq x \leq L$, so there is a constant λ such that

$$\frac{h''(y)}{h(y)} = \lambda = -\frac{\varphi''(x)}{\varphi(x)} \tag{8}$$

for all y with $0 \leq y \leq H$ and x with $0 \leq x \leq L$. From (8) we then have

$$-\varphi''(x) = \lambda\varphi(x) \text{ for all } x \text{ in } [0,L] \tag{9}$$

and

$$h''(y) = \lambda h(y) \text{ for all } y \text{ in } [0,H]. \tag{10}$$

It is worth noting that if

$$u(x,y) = \varphi(x)h(y)$$

and (9) and (10) hold, then

$$\begin{aligned}\frac{\partial^2 u}{\partial x^2}(x,y) &= \varphi''(x)h(y) = -\lambda\varphi(x)h(y) \\ &= -\varphi(x)h''(y) = -\frac{\partial^2 u}{\partial y^2}(x,y)\end{aligned}$$

so the PDE (1)

$$\frac{\partial^2 u}{\partial x^2}(x,y) + \frac{\partial^2 u}{\partial y^2}(x.y) = 0$$

will be satisfied, and we no longer need to assume that $u(x,y) \neq 0$. Continuing with our assumption that

$$u(x,y) = \varphi(x)h(y)$$

we have from conditions (2) and (3) that either $h(y) = 0$ for all y in $[0, H]$ which we reject because of (4) or

$$a\varphi(0) + b\varphi'(0) = 0 \tag{11}$$

and

$$c\varphi(L) + d\varphi'(L) = 0 \tag{12}$$

which we must accept. In a similar way we have from (4) that

$$h(H) = 0 \tag{13}$$

Let

$$\{\lambda_k\}_{k=1}^{\infty} \text{ and } \{\varphi_k\}_{k=1}^{\infty}$$

be a proper listing of eigenvalues and eigenfunctions for(9), (11), and (12). The equation (10)

$$h''(y) = \lambda h(y)$$

is equivalent to

$$h''(y) - \lambda h(y) = 0. \tag{14}$$

When $\lambda > 0$ as it must be because all eigenvalues for the problem (9), (11), and (12) are positive, a linearly independent pair of solutions to (14) is the pair whose values at y are

$$\sinh\sqrt{\lambda}y \text{ and } \sinh\sqrt{\lambda}(H-y).$$

Since h is a solution to (14), we have

$$h(y) = c_1 \sinh\sqrt{\lambda}y + c_2 \sinh\sqrt{\lambda}(H-y).$$

We have from (13) that $h(H) = 0$, so

$$c_1 \sinh\lambda H + c_2 \sinh\sqrt{\lambda}\cdot 0 = 0,$$

Using the fact that $\sinh 0 = 0$ and $\sinh z \neq 0$ when $z \neq 0$, we have that $c_1 = 0$ and see that when $\lambda = \lambda_k$ then the solutions to (13) and (14) are constant multiples of h_k where

$$h_k(y) = \sinh \sqrt{\lambda_k}(H - y).$$

The problem consisting of (1), (2), (3), and (4) is linear and homogeneous, so if $\{E_k\}_{k=1}^{n}$is a finite sequence of numbers and

$$u(x, y) = \sum_{k=1}^{n} E_k \varphi_k(x) h_k(y),$$

then u will be a solution to (1), (2), (3), and (4). Thus we hope that the solution to the problem consisting of (1) through (5) will be of the form

$$u(x, y) = \sum_{k=1}^{\infty} E_k \varphi_k(x) h_k(y)$$

for some perhaps infinite sequence of constants $\{E_k\}_{k=1}^{\infty}$.Condition (5)

$$u(x, 0) = f(x) \text{ for } x \text{ in } [0, L],$$

implies

$$f = \sum_{k=1}^{\infty} E_k \varphi_k h_k(0) = \sum_{k=1}^{\infty} (E_k \sinh \sqrt{\lambda_k} H) \varphi_k.$$

Since $\{\varphi_k\}_{k=1}^{\infty}$ is an orthogonal sequence of non zero function this implies

$$(E_k \sinh \sqrt{\lambda_k} H) = \frac{< f, \varphi_k >}{< \varphi_k, \varphi_k >}$$

so

$$E_k = \frac{< f, \varphi_k >}{\sinh \sqrt{\lambda_k} H < \varphi_k, \varphi_k >}$$

for $k = 1, 2, \ldots$In summary, the solution to the original problem (1) through (5) is u where

$$u(x, y) = \sum_{k=1}^{\infty} E_k \varphi_k(x) \sinh \sqrt{\lambda_k}(H - y)$$

in which

$$E_k = \frac{\int_0^L f(x) \varphi_k(x) dx}{\sinh \sqrt{\lambda_k} H \int_0^L (\varphi_k(x))^2 dx} \text{ for } k = 1, 2, \ldots$$

7. Suppose that each of α, β, γ, and δ is a real number, at least one of α and β is not zero and at least one of γ and δ is not zero. Suppose that each of c and L is a positive

number. Derive the solution to

$$\begin{aligned}
\frac{\partial^2 u}{\partial t^2}(x,t) &= c^2 \frac{\partial^2 u}{\partial x^2}(x,t) \text{ for } 0 \le x \le L \text{ and all } t \text{ in } \mathbb{R}, && (1) \\
\alpha u(0,t) + \beta \frac{\partial u}{\partial x}(0,t) &= 0 \text{ for all } t \text{ in } \mathbb{R}, && (2) \\
\gamma u(L,t) + \delta \frac{\partial u}{\partial x}(L,t) &= 0 \text{ for all } t \text{ in } \mathbb{R}, && (3) \\
u(x,0) &= f(x) \text{ for } 0 \le x \le L, \text{ and} && (4) \\
\frac{\partial u}{\partial t}(x,0) &= g(x) \text{ for } 0 \le x \le L. && (5)
\end{aligned}$$

Let $\{\lambda_k\}_{k=1}^{\infty}$ and $\{\varphi_k\}_{k=1}^{\infty}$ be a proper listing of eigenvalues and eigenfunctions for the related Sturm-Liouville problem. Suppose that all of the eigenvalues are positive.

Solution. Suppose that u is an elementary separated solution to (1). This means

$$u(x,t) = \varphi(x)h(t)$$

for some pair of one-place functions φ and h. Inserting this into (1), we have

$$\varphi(x)h''(t) = c^2 \varphi''(x)h(t). \tag{6}$$

Assuming for now that

$$u(x.t) \neq 0,$$

and dividing each side of (6) by $\varphi(x)h(t)$, we have

$$\frac{\varphi(x)h''(t)}{\varphi(x)h(t)} = c^2 \frac{\varphi''(x)h(t)}{\varphi(x)h(t)},$$

so

$$\frac{h''(t)}{h(t)} = c^2 \frac{\varphi''(x)}{\varphi(x)}.$$

This holds for all t and all x with $0 \le x \le L$, so there is a constant K such that

$$\frac{h''(t)}{h(t)} = K = c^2 \frac{\varphi''(x)}{\varphi(x)} \tag{7}$$

for all t and all x with $0 \le x \le L$. As a matter of notational convenience and so that we can more easily make use of our earlier work on two-point boundary value problems, we let

$$\lambda = -\frac{K}{c^2} \text{ so } K = -c^2\lambda.$$

From (7) we then have

$$-\varphi''(x) = \lambda \varphi(x) \text{ for all } x \text{ in } [0, L] \tag{8}$$

and

$$h''(t) = -\lambda c^2 h(t) \text{ for all } t. \tag{9}$$

It is worth noting that if

$$u(x,t) = \varphi(x)h(t)$$

and (8) and (9) hold, then

$$\begin{aligned}\frac{\partial^2 u}{\partial t^2}(x,t) &= \varphi(x)h''(t) = -\lambda c^2\varphi(x)h(t) \\ &= c^2\varphi''(x)h(t) = c^2\frac{\partial^2 u}{\partial x^2}(x,t)\end{aligned}$$

so the PDE (1)

$$\frac{\partial^2 u}{\partial t^2}(x,t) = c^2\frac{\partial^2 u}{\partial x^2}(x,t)$$

will be satisfied, and we no longer need to assume that $u(x,t) \neq 0$. Continuing with our assumption that

$$u(x,t) = \varphi(x)h(t)$$

We have from conditions (2) and (3)

$$\alpha\varphi(0) + \beta\varphi'(0) = 0 \tag{10}$$

and

$$\gamma\varphi(L) + \delta\varphi'(L) = 0. \tag{11}$$

Let

$$\{\lambda_k\}_{k=1}^{\infty} \text{ and } \{\varphi_k\}_{k=1}^{\infty}$$

be a proper listing for (8), (10), and (11). The equation (9)

$$h''(t) = -c^2\lambda h(t)$$

is equivalent to

$$h''(t) + c^2\lambda h(t) = 0. \tag{12}$$

When $\lambda > 0$ as it must be because all eigenvalues for the problem (8), (10), and (11) are positive, a linearly independent pair of solutions to (12) is the pair whose values at t are

$$\cos\sqrt{\lambda}ct \text{ and } \sin\sqrt{\lambda}ct.$$

Thus when $\lambda = \lambda_k$ the solutions to (9) are linear combinations of the functions h_{1k} and h_{2k} where

$$h_{1k}(t) = \cos\sqrt{\lambda_k}ct \text{ and } h_{2k}(t) = \sin\sqrt{\lambda_k}ct.$$

We expect that the solution to the problem consisting of (1) through (5) will be of the form

$$u(x,t) = \sum_{k=1}^{\infty}\varphi_k(x)[A_k h_{1k}(t) + B_k h_{2k}(t)] \tag{13}$$

for some sequences of constants $\{A_k\}_{k=1}^{\infty}$ and $\{B_k\}_{k=1}^{\infty}$.

Condition (4)

$$u(x,0) = f(x) \text{ for } x \text{ in } [0,L],$$

implies

$$f = \sum_{k=1}^{\infty} \varphi_k [A_k h_{1k}(0) + B_k h_{2k}(0)] = \sum_{k=1}^{\infty} [A_k \cos 0 + B_k \sin 0]\varphi_k = \sum_{k=1}^{\infty} A_k \varphi_k.$$

Since $\{\varphi_k\}_{k=1}^{\infty}$ is an orthogonal sequence of non zero functions this implies

$$A_k = \frac{< f, \varphi_k >}{< \varphi_k, \varphi_k >}$$

so for $k = 1, 2, \ldots$Returning to (13) we expect

$$\frac{\partial u}{\partial t}(x,t) = \sum_{k=1}^{\infty} \varphi_k(x)[A_k h'_{1k}(t) + B_k h'_{2k}(t)].$$

Condition (5)

$$\frac{\partial u}{\partial t}(x,0) = g(x) \text{ for all } x \text{ in } [0, L]$$

implies

$$g = \sum_{k=1}^{\infty} \varphi_k [A_k h'_{1k}(0) + B_k h'_{2k}(0)] = \sum_{k=1}^{\infty} (\sqrt{\lambda_k} c)[-A_k \sin 0 + B_k \cos 0]\varphi_k = \sum_{k=1}^{\infty} (\sqrt{\lambda_k} c) B_k \varphi_k$$

so

$$(\sqrt{\lambda_k} c) B_k = \frac{< g, \varphi_k >}{< \varphi_k, \varphi_k >} \text{ or } B_k = \frac{< g, \varphi_k >}{\sqrt{\lambda_k} c < \varphi_k, \varphi_k >}$$

for $k = 1, 2, 3, \ldots$.In summary, the solution to the original problem (1) through (5) is u where

$$u(x,t) = \sum_{k=1}^{\infty} [A_k \cos \frac{k\pi}{L} ct + B_k \sin \frac{k\pi}{L} ct]\varphi_k(x)$$

in which

$$A_k = \frac{< f, \varphi_k >}{< \varphi_k, \varphi_k >}$$

and

$$B_k = \frac{< g, \varphi_k >}{\sqrt{\lambda_k} c < \varphi_k, \varphi_k >} \text{ for } k = 1, 2, \ldots$$

8. Consider the following problem for Laplace's equation in a rectangle.

$$\frac{\partial^2 u}{\partial x^2}(x,y) + \frac{\partial^2 u}{\partial y^2}(x,y) = 0 \tag{1}$$

for all (x, y) in the rectangle $[0, 4] \times [0, 2]$ which is all (x, y) where $0 \le x \le 4$ and $0 \le y \le 2$ and

$$u(x, y) = B(x, y) \tag{2}$$

for all (x, y) on the boundary of $[0, 4] \times [0, 2]$ where

$$B(x, y) = x^2 + y^3 + xy + 1.$$

Find the function v of the form

$$v(x, y) = ax + by + cxy + d$$

such that

$$v(x, y) = B(x, y)$$

at each of the four corners of the rectangle. Then let w be given by

$$w(x, y) = u(x, y) - v(x, y)$$

for all (x, y) in the rectangle $[0, 4] \times [0, 2]$. Complete but **do not solve** the following problem statement for w.

$$\frac{\partial^2 w}{\partial x^2}(x, y) + \frac{\partial^2 w}{\partial y^2}(x, y) = ?$$

for all (x, y) in the rectangle $[0, 4] \times [0, 2]$,

$$w(x, 0) = ?$$

for $0 \le x \le 4$,

$$w(x, 2) = ?$$

for $0 \le x \le 4$,

$$w(0, y) = ?$$

for $0 \le y \le 2$, and

$$w(4, y) = ?$$

for $0 \le y \le 2$.

Solution.

$$d = v(0, 0) = B(0, 0) = 1$$

$$4a + d = v(4, 0) = B(4, 0) = 17$$

$$4a = 16$$

$$a = 4$$

$$2b + d = v(0, 2) = B(0, 2) = 9$$

$$2b = 8$$

$$b = 4$$

$$4 \cdot 4 + 4 \cdot 2 + 8c + 1 = v(4, 2) = B(4, 2) = 33$$

$$c = 1$$

$$v(x, y) = 4x + 4y + xy + 1$$

$$\begin{aligned}\frac{\partial^2 w}{\partial x^2}(x,y)+\frac{\partial^2 w}{\partial y^2}(x,y) &= (\frac{\partial^2 u}{\partial x^2}(x,y)+\frac{\partial^2 u}{\partial y^2}(x,y))-(\frac{\partial^2 v}{\partial x^2}(x,y)+\frac{\partial^2 v}{\partial y^2}(x,y)) \\ &= 0-0=0\end{aligned}$$

$$\begin{aligned}w(x,0) &= u(x,0)-v(x,0)=B(x,0)-v(x,0) \\ &= x^2-4x\end{aligned}$$

$$\begin{aligned}w(x,2) &= u(x,2)-v(x,2)=B(x,2)-v(x,2) \\ &= x^2-4x\end{aligned}$$

$$\begin{aligned}w(0,y) &= u(0,y)-v(0,y)=B(0,y)-v(0,y) \\ &= y^3-4y\end{aligned}$$

$$\begin{aligned}w(4,y) &= u(4,y)-v(4,y)=B(4,y)-v(4,y) \\ &= y^3-4y\end{aligned}$$

9. Let f be given by

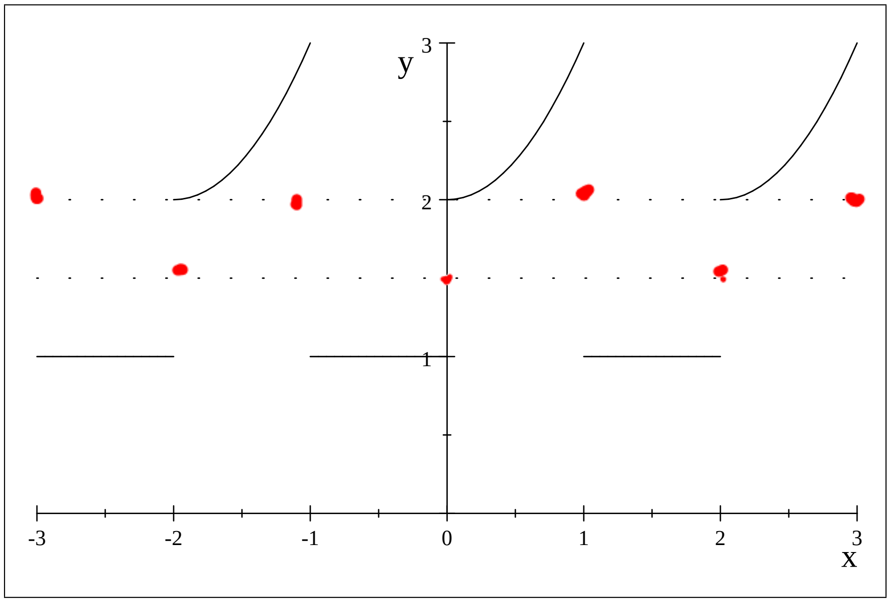
$$f(x)=\left\{\begin{array}{ccc} 1 & if & -1\le x<0 \\ 2+x^2 & if & 0\le x\le 1\end{array}\right. .$$

Let h be the limit of the Fourier Series ($L=1$) for f. Sketch the graph of h over $[-3,3]$. Be sure to show the value of h at each number in $[-3,3]$.

Solution.

$$\left\{\begin{array}{rcr} f(x+2) & if & -3<x<-1 \\ f(x) & if & -1<x<1 \\ f(x-2) & if & 1<x<3\end{array}\right.$$

$$1.5$$



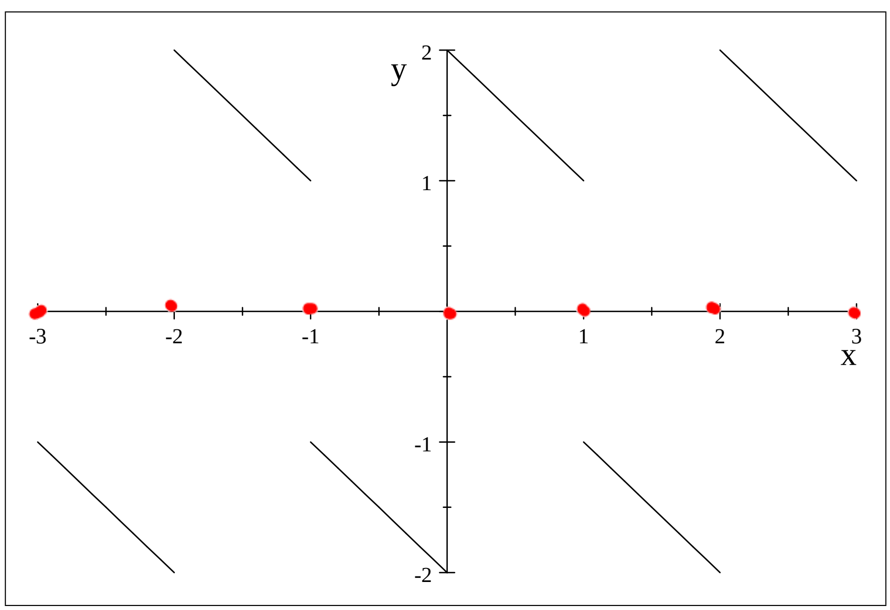
h

10. Let f be given by

$$f(x) = 2 - x \text{ for } 0 \le x \le 1.$$

(a) Let g be the limit of the sine series ($L = 1$) for f. Sketch the graph of g over $[-3, 3]$. Be sure to show the value of g at each number in $[-3, 3]$.

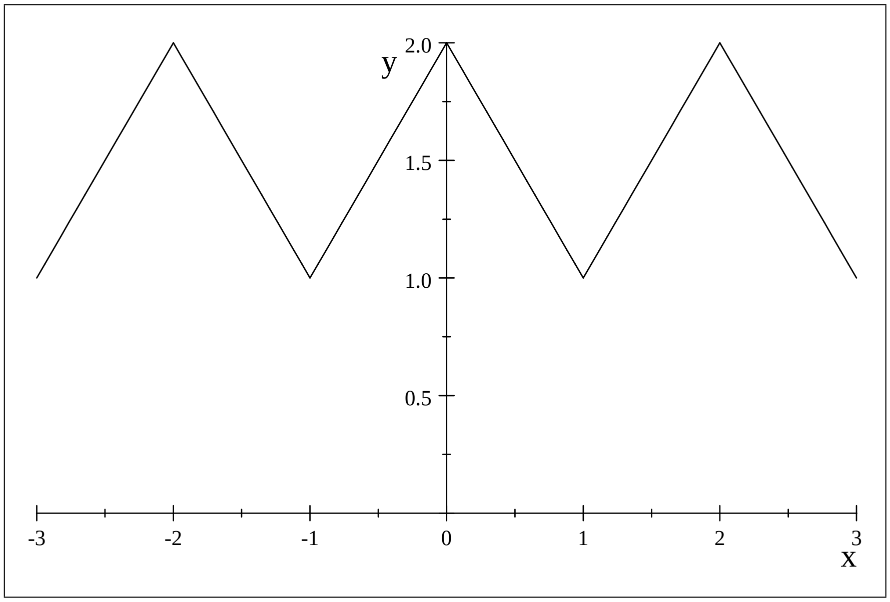
Solution.



g

(b) Let h be the limit of the cosine series ($L = 1$) for f. Sketch the graph of h over $[-3, 3]$. Be sure to show the value of h at each number in $[-3, 3]$.

Solution.



g

11. Suppose that f is a differentiable function with domain $\mathbb{R}$.

(a) Show that if f is an even function, then f' is an odd function.

Solution. If f is even

$$f(x) = f(-x)$$

so by the chain rule,

$$f'(x) = f'(-x)(-1) = -f'(-x).$$

Thus f' is odd.

(b) Show that if f is an odd function , then f' is an even function.

Solution. If f is odd

$$f(x) = -f(-x)$$

so by the chain rule,

$$f'(x) = -f'(-x)(-1) = f'(-x).$$

Thus f' is even.

12. Find the solution to

$$\begin{aligned}
\frac{\partial \varphi}{\partial t}(x,t) &= \frac{\partial^2 \varphi}{\partial x^2}(x,t) + xt \text{ for } 0 \le x \le 1 \text{ and } t \ge 0, && (1)\\
\varphi(0,t) &= 0 \text{ for } t \ge 0, && (2)\\
\varphi(1,t) &= 0 \text{ for } t \ge 0, \text{and} && (3)\\
\varphi(x,0) &= \sin \pi x \text{ for } 0 \le x \le 1. && (4)
\end{aligned}$$

Solution. The related Sturm-Liouville problem is

$$-\psi''(x) = \lambda \psi(x) \text{ for } 0 \le x \le 1,$$

$$\psi(0) = 0, \text{ and}$$
$$\psi(1) = 0.$$

A proper listing of eigenvalues and eigenfunctions is $\{\lambda_k\}_{k=1}^{\infty}$ and $\{\psi_k\}_{k=1}^{\infty}$ where

$$\lambda_k = (k\pi)^2 \text{ and } \psi_k(x) = \sin k\pi x.$$

Look for a solution to (1)-(4) in the form

$$u(x,t) = \sum_{k=1}^{\infty} b_k(t)\psi_k(x) \tag{5}$$

where the functions b_k are to be determined.

Putting (5) into (1) produces

$$\sum_{k=1}^{\infty} b_k'(t)\psi_k(x) = \left[\sum_{k=1}^{\infty} b_k(t)\psi_k''(x)\right] + xt.$$

Using

$$-\psi_k'' = \lambda_k \psi_k$$

this becomes

$$\sum_{k=1}^{\infty} b_k'(t)\psi_k(x) = \left[-\sum_{k=1}^{\infty} b_k(t)\lambda_k\psi_k(x)\right] + xt.$$

or

$$\sum_{k=1}^{\infty} \left(b_k'(t) + \lambda_k b_k(t)\right) \psi_k(x) = xt \tag{6}$$

Expanding xt in terms of $\{\psi_k\}$ for each t we have

$$xt = \sum_{k=1}^{\infty} \gamma_k(t)\psi_k(x)$$

where

$$\begin{aligned}
\gamma_k(t) &= \frac{\int_0^1 xt\psi_k(x)dx}{\int_0^1 |\psi_k(x)|^2 dx} = \frac{t\int_0^1 x\psi_k(x)dx}{\int_0^1 |\psi_k(x)|^2 dx} \\
&= \frac{t\int_0^1 x\sin k\pi x dx}{\int_0^1 (\sin k\pi x)^2 dx} = 2t\frac{(-1)^{k+1}}{k\pi}
\end{aligned}$$

From (6) we have

$$\sum_{k=1}^{\infty} \left(b_k'(t) + \lambda_k b_k(t)\right) \psi_k(x) = \sum_{k=1}^{\infty} \gamma_k(t)\psi_k(x). \tag{7}$$

Equating coefficients on ψ_k we have

$$b_k'(t) + \lambda_k b_k(t) = \gamma_k(t) \tag{8}$$

for $t \geq 0$ and $k = 1, 2, \ldots$. To complete the determination of each b_k we need an initial condition to go with (8). From (4) and (5) we have

$$f(x) = \sum_{k=1}^{\infty} b_k(0)\psi_k(x).$$

where, in this problem,

$$f(x) = \sin \pi x$$

Thus

$$b_k(0) = \frac{< f, \psi_k >}{< \psi_k, \psi_k >} = \begin{cases} 1 & if \quad k = 1 \\ 0 & if \quad k = 2, 3, \ldots \end{cases}.$$

The function b_1 is determined by

$$\begin{aligned}
b_1'(t) + \pi^2 b_1(t) &= -\frac{2t}{\pi} \\
b_1(0) &= 1.
\end{aligned}$$

Solving the first order linear initial value problem, we have

$$b_1(t) = \frac{1}{\pi^5}\left((\pi^5 - 2)e^{-\pi^2 t} + 2(1 - \pi^2 t)\right)$$

For $k = 2, 3, \ldots$, b_k is determined by

$$\begin{aligned} b_k'(t) + (k\pi)^2 b_k(t) &= 2t\frac{(-1)^{k+1}}{k\pi} \\ b_k(0) &= 0 \end{aligned}$$

Solving the first order linear initial value problem, we have

$$b_k(t) = \frac{2(-1)^{k+1}}{\pi^5 k^5}\left(e^{-\pi^2 k^2 t} - (1 - \pi^2 k^2 t)\right)$$

The solution φ to (1)-(4) is given by

$$\begin{aligned} \varphi(x,t) &= \frac{1}{\pi^5}\left((\pi^5 - 2)e^{-\pi^2 t} + 2(1 - \pi^2 t)\right)\sin \pi x \\ &\quad + \frac{2}{\pi^5}\sum_{k=2}^{\infty} \frac{(-1)^{k+1}}{k^5}\left(e^{-\pi^2 k^2 t} - (1 - \pi^2 k^2 t)\right)\sin k\pi x \end{aligned}$$

13. Suppose that each of L, H, and T is a positive number. Find the eigenvalues and eigenfunctions for

$$\begin{aligned} -\left(\frac{\partial^2 \psi}{\partial x^2}(x,y,z) + \frac{\partial^2 \psi}{\partial y^2}(x,y,z) + \frac{\partial^2 \psi}{\partial z^2}(x,y,z)\right) &= \lambda \psi(x,y,z) && (1) \\ \psi(0,y,z) &= 0 && (2) \\ \psi(L,y,z) &= 0 && (3) \\ \psi(x,0,z) &= 0 && (4) \\ \psi(x,H,z) &= 0 && (5) \\ \psi(x,y,0) &= 0 \text{ and} && (6) \\ \psi(x,y,T) &= 0 && (7) \end{aligned}$$

for $0 \le x \le L$, $0 \le y \le H$, and $0 \le z \le T$. Start by looking for elementary separated solutions of the form

$$\psi(x,y,z) = \varphi(x,y)h(z). \tag{8}$$

Solution. Suppose that

$$\psi(x,y,z) = \varphi(x,y)h(z). \tag{8}$$

From (1) it follows that

$$-\left(\frac{\partial^2 \varphi}{\partial x^2}(x,y) + \frac{\partial^2 \varphi}{\partial y^2}(x,y)\right)h(z) - \varphi(x,y)h''(z) = \lambda \varphi(x,y)h(z).$$

Dividing each side by $\varphi(x,y)h(z)$ (assuming for now that it is not zero) produces

$$-\frac{\frac{\partial^2 \varphi}{\partial x^2}(x,y) + \frac{\partial^2 \varphi}{\partial y^2}(x,y)}{\varphi(x,y)} = \lambda + \frac{h''(z)}{h(z)}.$$

Letting μ be the common constant value, we have

$$-\left(\frac{\partial^2\varphi}{\partial x^2}(x,y)+\frac{\partial^2\varphi}{\partial y^2}(x,y)\right)=\mu\varphi(x,y) \tag{9}$$

for $0 \le x \le L$ and $0 \le y \le H$ and

$$h''(z)=(\mu-\lambda)h(z)$$

or

$$-h''(z)=\delta h(z) \text{ for } 0 \le z \le T \tag{10}$$

where

$$\delta=\lambda-\mu.$$

Note that

$$\lambda=\mu+\delta. \tag{11}$$

From (8) and conditions (2)-(7) we have

$$\varphi(x,y)=0 \text{ for } (x,y) \text{ on the boundary of } [0,L]\times[0,H], \tag{12}$$

$$h(0)=0, \tag{13}$$

and

$$h(T)=0. \tag{14}$$

A proper listing of eigenvalues and eigenfunctions for (9) and (12) is $\{\mu_{kj}\}_{k,j=1}^{\infty}$ and $\{\varphi_{kj}\}_{k,j=1}^{\infty}$ where

$$\mu_{kj}=(\frac{k\pi}{L})^2+(\frac{j\pi}{H})^2 \text{ and } \varphi_{kj}(x,y)=\sin\frac{k\pi x}{L}\sin\frac{j\pi y}{H}.$$

A proper listing of eigenvalues and eigenfunctions for (10), (13), and (14) is $\{\delta_l\}_{l=1}^{\infty}$ and $\{h_l\}_{l=1}^{\infty}$ where

$$\delta_l=(\frac{l\pi}{T})^2 \text{ and } h_l(z)=\sin\frac{l\pi z}{T}.$$

In view of (8) and (11), a proper listing of eigenvalues and eigenfunctions for (1)-(7) is $\{\lambda_{kjl}\}_{k,j,l=1}^{\infty}$ and $\{\psi_{kjl}\}_{k,j,l=1}^{\infty}$ where

$$\lambda_{kjl}=\mu_{kj}+\delta_l=(\frac{k\pi}{L})^2+(\frac{j\pi}{H})^2+(\frac{l\pi}{T})^2$$

and

$$\psi_{kjl}(x,y,z)=\varphi_{kj}(x,y)h_l(z)=\sin\frac{k\pi x}{L}\sin\frac{j\pi y}{H}\sin\frac{l\pi z}{T}.$$

14. Solve the following differential equation which we encountered when deriving d'Alembert's solution to the wave equation.

$$\frac{\partial^2 w}{\partial t\partial x}(x,t)=0 \text{ for all } x \text{ and } t.$$

Solution. From

$$\frac{\partial^2 w}{\partial t \partial x}(x,t) = 0$$

we get

$$\frac{\partial w}{\partial x}(x,t) = h(x)$$

for some one-place function h. From this we get

$$w(x,t) = f(x) + g(t)$$

where f is an anti-derivative of h and g is a one-place function.

15. Let u be the solution to

$$\begin{aligned}
\frac{\partial^2 u}{\partial t^2}(x,t) &= \frac{\partial^2 u}{\partial x^2}(x,t) \text{ for all } x \text{ and } t \text{ in } \mathbb{R}, \\
u(x,0) &= \varphi(x) \text{ for all } x \text{ in } \mathbb{R}, \text{ and} \\
\frac{\partial u}{\partial t}(x,0) &= \psi(x) \text{ for all } x \text{ in } \mathbb{R}.
\end{aligned}$$

where

$$\varphi(x) = 0 \text{ for all } x \text{ and}$$

$$\psi(x) = \begin{cases} 0 & \text{for} \quad x < -1 \\ 2x+2 & \text{for} \quad -1 \le x \le 0 \\ 2-2x & \text{for} \quad 0 \le x \le 1 \\ 0 & \text{for} \quad x > 1 \end{cases}.$$

Let

$$h(x) = u(x,3) \text{ for all } x \text{ in } \mathbb{R}.$$

Sketch the graph of h on the interval $[-6,6]$.

Suggestion: Show that

$$u(x,t) = F(x+t) - F(x-t)$$

where

$$F(x) = \frac{1}{2}\int_0^x \psi(s)ds.$$

The graph of h is the graph of F shifted 3 units to the left plus the graph of $-F$ shifted 3 units to the right.

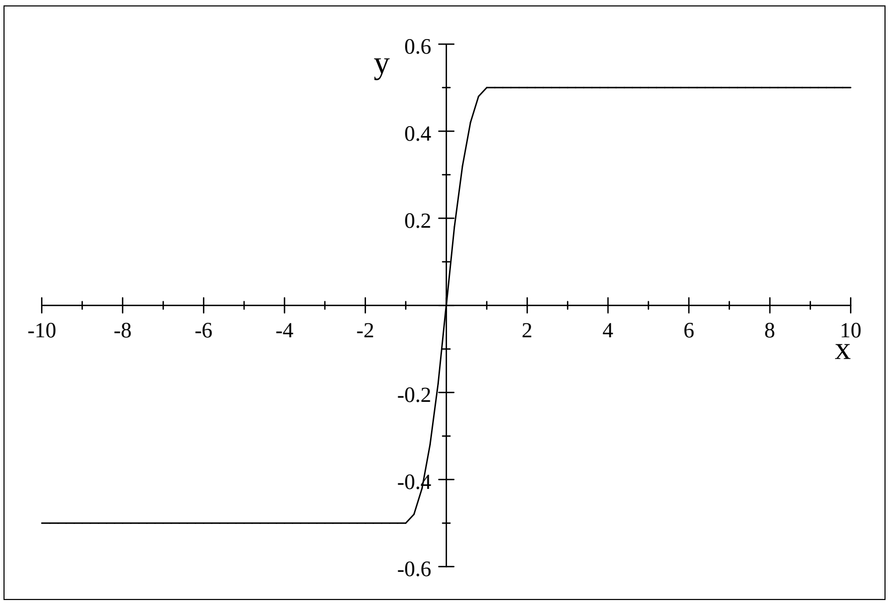
Solution.

$$
\begin{aligned}
u(x,t) &= \frac{1}{2}\int_{x-t}^{x+t}\psi(s)ds = \frac{1}{2}\int_{x-t}^{0}\psi(s)ds + \frac{1}{2}\int_{0}^{x+t}\psi(s)ds \\
&= \frac{1}{2}\int_{0}^{x+t}\psi(s)ds - \frac{1}{2}\int_{0}^{x-t}\psi(s)ds \\
&= F(x+t) - F(x-t)
\end{aligned}
$$

where

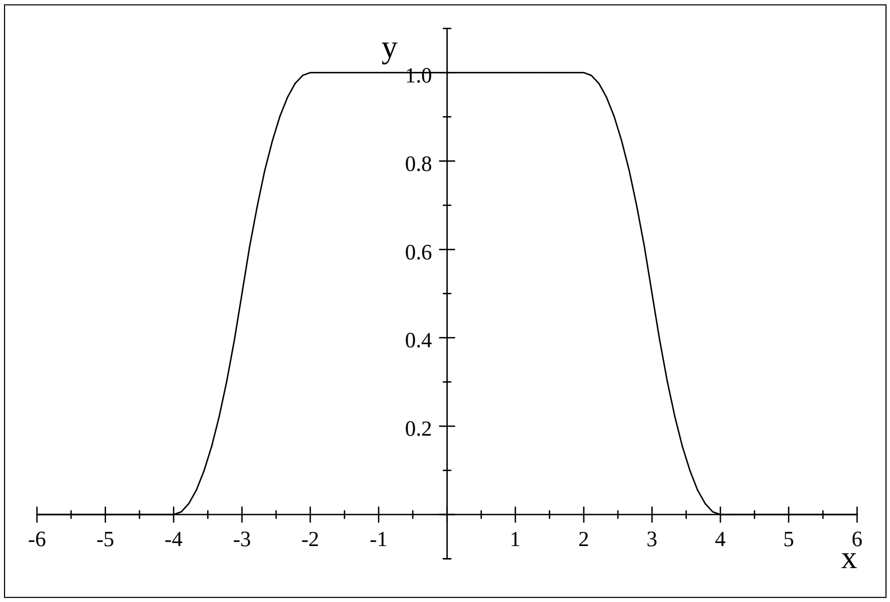
$$F(x) = \frac{1}{2}\int_0^x \psi(s)ds.$$

$$F(x) = \begin{cases} -\frac{1}{2} & if \quad x < -1 \\ \frac{1}{2}x^2 + x & if \quad -1 \le x < 0 \\ x - \frac{1}{2}x^2 & if \quad 0 \le x \le 1 \\ \frac{1}{2} & if \quad x > 1 \end{cases}$$



F

$$F(x+3) - F(x-3)$$



h

16. Find the solution to Laplace's equation in polar coordinates

$$\frac{\partial^2 u}{\partial r^2}(r,\theta) + \frac{1}{r}\frac{\partial u}{\partial r}(r,\theta) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}(r,\theta) = 0 \tag{1}$$

in the half-annulus where $2 \le r \le 4$ and $0 \le \theta \le \pi$ subject to

$$\begin{aligned} u(r,0) &= 0 \text{ for } 2 \le r \le 4 && (2)\\ u(r,\pi) &= 0 \text{ for } 2 \le r \le 4 && (3)\\ u(2,\theta) &= f(\theta) \text{ for } 0 \le \theta \le \pi \text{ and} && (4)\\ u(4,\theta) &= g(\theta) \text{ for } 0 \le \theta \le \pi. && (5) \end{aligned}$$

Solution. Suppose that u is an elementary separated solution to (1). This means

$$u(r,\theta) = \varphi(\theta)G(r)$$

for some pair of one-place functions φ and G. Inserting this into (1), we have

$$\varphi(\theta)G''(r) + \frac{1}{r}\varphi(\theta)G'(r) + \frac{1}{r^2}\varphi''(\theta)G(r) = 0 \tag{6}$$

Assuming for now that

$$u(r,\theta) \neq 0,$$

and dividing each side of (6) by

$$\frac{\varphi(\theta)G(r)}{r^2}$$

we have

$$r^2\frac{G''(r)}{G(r)} + r\frac{G'(r)}{G(r)} = -\frac{\varphi''(\theta)}{\varphi(\theta)}$$

This holds for all r with $2 \le r \le 4$ and θ with $0 \le \theta \le \pi$; so there is a constant λ such that

$$r^2\frac{G''(r)}{G(r)} + r\frac{G'(r)}{G(r)} = -\frac{\varphi''(\theta)}{\varphi(\theta)} = \lambda \tag{7}$$

for all r with $2 \le r \le 4$ and θ with $0 \le \theta \le \pi$. From (7) we then have

$$-\varphi''(\theta) = \lambda\varphi(\theta) \text{ for all } \theta \text{ in } [0, \pi] \tag{8}$$

and

$$r^2G''(r) + rG'(r) - \lambda G(r) = 0 \text{ for all } r \text{ in } [2, 4]. \tag{9}$$

If

$$u(r, \theta) = \varphi(\theta)G(r)$$

and (8) and (9) hold, then

$$\begin{aligned}
\frac{\partial^2 u}{\partial r^2}(r,\theta) + \frac{1}{r}\frac{\partial u}{\partial r}(r,\theta) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}(r,\theta) &= \varphi(\theta)G''(r) + \frac{1}{r}\varphi(\theta)G'(r) + \frac{1}{r^2}\varphi''(\theta)G(r) \\
&= \varphi(\theta)\left(G''(r) + \frac{1}{r}G'(r)\right) + \left(\frac{1}{r^2}\varphi''(\theta)G(r)\right) \\
&= \varphi(\theta)\lambda\frac{1}{r^2}G(r) - \frac{1}{r^2}\lambda\varphi(\theta)G(r) \\
&= 0
\end{aligned}$$

so the PDE (1) will be satisfied, and we no longer need to assume that $u(r, \theta) \neq 0$.

Continuing with our assumption that

$$u(r, \theta) = \varphi(\theta)G(r)$$

we have from conditions (2) and (3) that either $G(r) = 0$ for all r in $[2, 4]$ which we reject because of (5) and (4) or

$$\varphi(0) = 0 \tag{10}$$

and

$$\varphi(\pi) = 0 \tag{11}$$

which we then must accept.
The two-point boundary value problem consisting of (8), (10), and (11) is one which we have studied. A proper listing of eigenvalues and eigenfunctions for this problem is $\{\lambda_k\}_{k=1}^{\infty}$ and $\{\varphi_k\}_{k=1}^{\infty}$ where

$$\begin{aligned}
\lambda_k &= k^2 \text{ for } k = 1, 2, 3, \ldots, \text{ and} \\
\varphi_k(\theta) &= \sin k\theta \text{ for } k = 1, 2, 3, \ldots \text{ and } 0 \le \theta \le \pi.
\end{aligned}$$

Equation (9) is a Cauchy-Euler equation. When k is a positive integer and $\lambda = k^2$ then

$$G(r) = c_1r^k + c_2r^{-k}.$$

Considering the possible combinations, we expect the solution to (1)-(5) to be of the form

$$u(r,\theta) = \sum_{k=1}^{\infty} \left(A_k r^k + B_k r^{-k}\right) \sin k\theta. \tag{12}$$

for $2 \le r \le 4$ and $0 \le \theta \le \pi$.
In order that (4) and (5) hold it is necessary and sufficient that

$$f(\theta) = \sum_{k=1}^{\infty} \left[\left(A_k 2^k + B_k 2^{-k}\right) \sin k\theta\right] \tag{13}$$

and

$$g(\theta) = \sum_{k=1}^{\infty} \left(A_k 4^k + B_k 4^{-k}\right) \sin k\theta \tag{14}$$

for $0 \le \theta \le \pi$. Thus

$$A_k 2^k + B_k 2^{-k} = \frac{2}{\pi} \int_0^{\pi} f(\theta) \sin k\theta d\theta, \tag{15}$$

and

$$A_k 4^k + B_k 4^{-k} = \frac{2}{\pi} \int_0^{\pi} g(\theta) \sin k\theta d\theta. \tag{16}$$

The solution is given by (12) where for each k, the coefficients A_k and B_k are determined by (15) and (16).

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1. Heat is flowing in a region in the plane with flux $\boldsymbol{\varphi}$ given by

$$\boldsymbol{\varphi}(x,y) = (1+xy, 1+x^2y) = (1+xy)\mathbf{i} + (1+x^2y)\mathbf{j}.$$

The rectangle R which consists of all (x,y) where $0 \le x \le 4$ and $0 \le y \le 2$ is contained in this region. There are no sinks or sources in R. Find the rate at which the total heat energy in R is changing.

Solution. The rate of change of heat energy in R is

$$-\oint_C \boldsymbol{\varphi} \cdot \mathbf{n} ds$$

where C is the boundary of the rectangle traversed once in the positive direction and $\mathbf{n}$ is the outward unit normal vector. The divergence theorem in the plane tell us that this is the same as

$$-\iint_R \nabla \cdot \boldsymbol{\varphi} dA$$

$$-\iint_R \nabla \cdot \boldsymbol{\varphi} dA = -\int_{x=0}^{x=4} \int_{y=0}^{y=2} (y+x^2) dy dx = -\frac{152}{3}$$