Math 3363 Spring 2020

Homework 1 Solutions

Please use a pencil and do the problems in the order in which they are listed. Write on only one side of each page and staple your pages.

1. Do Problem 1.4.1(b) on page 18. The problem to solve is

$$\begin{aligned} u''(x) &= 0 \text{ for } 0 \leq x \leq L, \\ u(0) &= T, \text{and} \\ u(L) &= 0. \end{aligned}$$

Solution.

$$u(x) = C_1 x + C_2.$$

$u(0) = T$ so $C_2 = T$. Then $0 = C_1 \cdot L + T$ so $C_1 = -\frac{T}{L}$.

$$u(x) = -\frac{T}{L}x + T$$

2. Do Problem 1.4.1(e) on page 18. The problem to solve is

$$\begin{aligned} K_0 u''(x) + Q &= 0 \text{ for } 0 \leq x \leq L, \\ u(0) &= T_1 \text{ and} \\ u(L) &= T_2 \end{aligned}$$

when

$$\frac{Q}{K_0} = 1.$$

Solution.

$$u''(x) = -\frac{Q}{K_0}$$

so

$$u''(x) = -1.$$

From this we get

$$u(x) = -\frac{1}{2}x^2 + C_1 x + C_2.$$

$u(0) = T_1$ implies $C_2 = T_1$. Then $u(L) = T_2$ implies $T_2 = -\frac{1}{2}L^2 + C_1 L + T_1$ implying $C_1 = \frac{T_2 - T_1}{L} + \frac{1}{2}L$. Thus

$$u(x) = -\frac{1}{2}x^2 + (\frac{T_2 - T_1}{L} + \frac{1}{2}L)x + T_1.$$

3. Do Problem 1.4.1(g) on page 18. The problem to solve is

$$\begin{aligned} u''(x) &= 0 \text{ for } 0 \leq x \leq L, \\ u(0) &= T, \text{and} \\ u'(L) + u(L) &= 0. \end{aligned}$$

Solution.

$$u(x) = C_1 x + C_{2.}.$$

The endpoint conditions imply $C_2 = T$ then $C_1 + C_1 L + T + 0$. So

$$u(x) = \frac{-T}{1+L} x + T.$$

4. Do Problem 1.4.2(a) on page 18 of the text. You are being told that $Q(x) = K_0 x$ is the heat per unit time per unit length being generated at coordinate x for $0 \leq x \leq L$.

Solution. The heat per unit time being generated in the entire rod is

$$\int_0^L Q(x)dx = \int_0^L K_0 x dx = \frac{1}{2} K_0 L^2.$$

5. Do Problem 1.4.2(b) on page 18 of the text. The differential equation to solve is

$$K_0 u''(x) + Q(x) = 0.$$

K_0 is a constant and heat flowing to the right at coordinate x is $-K_0 u'(x)$. Note that heat flowing out at the left end of the rod is $-(-K_0 u'(0)) = K_0 u'(0)$.

Solution. Since $Q(x)/K_0 = x$, we have

$$u''(x) = -x, \ u(0) = 0, \text{ and } u(L) = 0.$$

Thus, integrating twice and using the end point conditions,

$$u(x) = -\frac{1}{6} x^3 + \frac{1}{6} L^2 x$$

so

$$u'(x) = -\frac{1}{2} x^2 + \frac{1}{6} L^2.$$

Heat energy flowing to the right per unit time at coordinate x is

$$-K_0 u'(x) = -K_0 (-\frac{1}{2} x^2 + \frac{1}{6} L^2).$$

At $x = 0$,heat flowing out is

$$K_0 u'(0) = \frac{1}{6} K_0 L^2.$$

At $x = L$,heat flowing out is

$$-K_0(L) = -K_0(-\frac{1}{2}L^2 + \frac{1}{6}L^2) = \frac{1}{3}K_0 L^2.$$

6. Do Problem 1.4.2(c) on page 18 of the text.

Solution. Per unit time, the heat energy being generated inside the rod is the same as the total heat energy flowing out of the rod. This is the case since

$$\frac{1}{2}K_0 L^2 = \frac{1}{6}K_0 L^2 + \frac{1}{3}K_0 L^2.$$

7. Let

$$Q(x) = 10x - x^2$$

and let

$$f(x) = |x - 5| - 5 \text{ for } 0 \leq x \leq 10.$$

Find the value of β so that the following problem has an equilibrium solution.

$$\frac{\partial w}{\partial t}(x,t) = \frac{\partial^2 w}{\partial x^2}(x,t) + Q(x) \text{ for } 0 \leq x \leq 10 \text{ and } t \geq 0,$$

$$\frac{\partial w}{\partial x}(0,t) = \beta \text{ for } t \geq 0 \ ,$$

$$\frac{\partial w}{\partial x}(10,t) = 1 \text{ for } t \geq 0, \text{ and}$$

$$w(x,0) = f(x) \text{ for } 0 \leq x \leq 10.$$

The related equilibrium problem is stated below in Problem 8.

Solution. We have

$$u''(x) = x^2 - 10x$$

so

$$\int_0^{10} u''(x)dx = \int_0^{10} (x^2 - 10x)dx$$

from which we get

$$u'(10) - u'(0) = -\frac{500}{3} \text{ or } 1 - \beta = -\frac{500}{3}.$$

So

$$\beta = \frac{503}{3}.$$

8. Find the equilibrium solution u for Problem 7. The problem for u is

$$0 = u''(x) + Q(x) \text{ for } 0 \le x \le 10,$$

$$u'(0) = \beta, \text{ and}$$

$$u'(10) = 1.$$

Use the value for β that you found in problem 7. There should be no undetermined constant in your answer. You may assume that

$$\lim_{t\to\infty} w(x,t) = u(x)$$

for $0 \le x \le 10$, and you may interchange limits and integrals when you need to do this.

Solution. Starting with

$$u''(x) = x^2 - 10x$$

we have

$$u'(x) = \frac{1}{3}x^3 - 5x^2 + c_1.$$

Since $u'(0) = \beta$, it follows that $c_1 = \beta = \dfrac{503}{3}$ so

$$u(x) = \frac{1}{12}x^4 - \frac{5}{3}x^3 + \frac{503}{3}x + c_2.$$

We need to find c_2. To do this, we first show that

$$\int_0^{10} w(x,t)dx$$

is constant in time. We expect this because the total thermal energy in the rod,

$$\int_0^{10} e(x,t)dx,$$

should be constant and

$$e(x,t) = c\rho(w(x,t) - Z).$$

$$\begin{aligned}
\frac{d}{dt}\int_0^{10} w(x,t)dx &= \int_0^{10} \frac{\partial w}{\partial t}(x,t)dx = \int_0^{10} \left(\frac{\partial^2 w}{\partial x^2}(x,t) + Q(x)\right) dx \\
&= \frac{\partial w}{\partial x}(10,t) - \frac{\partial w}{\partial x}(0,t) + \int_0^{10} (10x - x^2)dx \\
&= 1 - \beta + \frac{500}{3} = 1 - \frac{503}{3} + \frac{500}{3} \\
&= 0.
\end{aligned}$$

Thus

$$\int_0^{10} w(x,t)dx$$

is constant in t.

$$\begin{aligned}
\int_0^{10} f(x)dx &= \int_0^{10} w(x,0)dx \\
&= \int_0^{10} w(x,t)dx \text{ (any } t\text{)} \\
&= \lim_{t\to\infty}\int_0^{10} w(x,t)dx = \int_0^{10} \lim_{t\to\infty} w(x,t)dx \\
&= \int_0^{10} u(x)dx = \int_0^{10} \left(\frac{1}{12}x^4 - \frac{5}{3}x^3 + \frac{503}{3}x + c_2\right)dx \\
&= \frac{17\,650}{3} + 10c_2
\end{aligned}$$

Thus

$$c_2 = -\frac{17\,65}{3} + \frac{1}{10}\int_0^{10} f(x)dx.$$

$$\int_0^{10} f(x)dx = \int_0^5 (-x)dx + \int_5^{10} (x-10)dx = -25$$

so

$$c_2 = -\frac{17\,65}{3} - \frac{25}{10} = -\frac{3545}{6}$$

and

$$u(x) = \frac{1}{12}x^4 - \frac{5}{3}x^3 + \frac{503}{3}x - \frac{35\,45}{6}$$