Math 3363 Spring 2020

Homework 2 Solutions

Please use a pencil and do the problems in the order in which they are listed. Write on only one side of each page and staple your pages.

1. Use the Rayleigh Quotient to show that all eigenvalues for

$$\begin{aligned} -\varphi''(x) &= \lambda\varphi(x) \text{ for } 0 \le x \le L, \quad (1)\\ \varphi'(0) &= 0, \text{ and} \quad (2)\\ \varphi(L) &= 0. \quad (3) \end{aligned}$$

are non-negative. Note that L is a positive number.

Solution. If λ is an eigenvalue and φ is a corresponding eigenfunction,

$$\begin{aligned} \lambda &= \frac{\varphi(0)\varphi'(0) - \varphi(L)\varphi'(L) + \int_0^L (\varphi'(x))^2 dx}{\int_0^L (\varphi(x))^2 dx} \\ &= \frac{\varphi(0)\cdot 0 - 0\cdot\varphi'(L) + \int_0^L (\varphi'(x))^2 dx}{\int_0^L (\varphi(x))^2 dx} \\ &= \frac{\int_0^L (\varphi'(x))^2 dx}{\int_0^L (\varphi(x))^2 dx} \ge 0 \end{aligned}$$

so all eigenvalues are nonnegative.

2. Show that zero is not an eigenvalue for (1), (2), and (3).

Solution. We will show that the only solution to (1), (2), and (3) when $\lambda = 0$ is given by $\varphi(x) = 0$.Thus 0 is not an eigenvalue.

If $\lambda = 0$ and φ is a solution to (1) then $\varphi(x) = c_1 + c_2 x$ and $\varphi'(x) = c_2$. From (2) it follows that $c_2 = 0$ then from (3) it follows that $c_1 = 0$ so $\varphi(x) = 0$ for $0 \le x \le L$.

3. Find the matrix $D(\lambda)$ for (1), (2), and (3) when $\lambda > 0$.Note that (2) and (3) are equivalent to

$$\begin{aligned}\beta_1\varphi(0)+\beta_2\varphi'(0) &= 0 \text{ and}\\ \beta_3\varphi(L)+\beta_4\varphi'(L) &= 0\end{aligned}$$

where $\beta_1 = 0$, $\beta_2 = 1$, $\beta_3 = 1$, and $\beta_4 = 0$.

Solution.

$$D(\lambda) = \begin{pmatrix} \beta_1 & \beta_2 \\ 0 & 0 \end{pmatrix} \Phi_\lambda(0) + \begin{pmatrix} 0 & 0 \\ \beta_3 & \beta_4 \end{pmatrix} \Phi_\lambda(L)$$

$$\Phi_\lambda(x) = \begin{pmatrix} \cos\sqrt{\lambda}x & \sin\sqrt{\lambda}x \\ -\sqrt{\lambda}\sin\sqrt{\lambda}x & \sqrt{\lambda}\cos\sqrt{\lambda}x \end{pmatrix}$$

$$\begin{aligned} D(\lambda) &= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \Phi_\lambda(0) + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \Phi_\lambda(L) \\ &= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & \sqrt{\lambda} \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \cos\sqrt{\lambda}L & \sin\sqrt{\lambda}L \\ -\sqrt{\lambda}\sin\sqrt{\lambda}L & \sqrt{\lambda}\cos\sqrt{\lambda}L \end{pmatrix} \\ &= \begin{pmatrix} 0 & \sqrt{\lambda} \\ \cos\sqrt{\lambda}L & \sin\sqrt{\lambda}L \end{pmatrix} \end{aligned}$$

4. Find the function $\Delta(\lambda)$ for (1), (2), and (3) when $\lambda > 0$ then find the eigenvalues.

Solution.

$$\Delta(\lambda) = \det D(\lambda) = -\sqrt{\lambda}\cos\sqrt{\lambda}L$$

λ is an eigenvalue if and only if $\lambda > 0$ and

$$-\sqrt{\lambda}\cos\sqrt{\lambda}L = 0$$

if and only if $\lambda > 0$ and

$$\cos\sqrt{\lambda}L = 0$$

if and only if

$$\sqrt{\lambda}L = (k-\frac{1}{2})\pi \text{ or } \sqrt{\lambda} = \frac{(2k-1)\pi}{2L}$$

or

$$\lambda = \left(\frac{(2k-1)\pi}{2L}\right)^2$$

for some positive integer k.

5. For each eigenvalue for (1), (2), and (3), find the corresponding eigenfunctions.

Solution. When λ_0 is an eigenvalue, φ is a corresponding eigenfunction if and only if

$$\varphi(x) = c_1 \cos \sqrt{\lambda_0 x} + c_2 \sin \sqrt{\lambda_0} x$$

for some pair of numbers c_1 and c_2 at least one of which is not zero and

$$D(\lambda_0) \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Here

$$\begin{aligned} D\left(\left(\frac{(2k-1)\pi}{2L}\right)^2\right) &= \begin{pmatrix} 0 & \frac{(2k-1)\pi}{2L} \\ \cos \frac{(2k-1)\pi}{2} & \sin \frac{(2k-1)\pi}{2} \end{pmatrix} \\ &= \begin{pmatrix} 0 & \frac{(2k-1)\pi}{2L} \\ 0 & (-1)^{k+1} \end{pmatrix} \end{aligned}$$

so

$$D\left(\left(\frac{(2k-1)\pi}{2L}\right)^2\right) \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

if and only if $c_2 = 0$. Thus φ is an eigenfunction corresponding to the eigenvalue $\left(\frac{(2k-1)\pi}{2L}\right)^2$ if and only

$$\varphi(x) = c_1 \cos \frac{(2k-1)\pi x}{2L}$$

for some $c_1 \neq 0$.

6. A proper listing of eigenvalues and eigenfunctions for

$$\begin{aligned} -\varphi''(x) &= \lambda\varphi(x) \text{ for } 0 \leq x \leq L, \\ \varphi'(0) &= 0, \text{ and} \\ \varphi'(L) &= 0 \end{aligned}$$

is $\{\lambda_k\}_{k=0}^{\infty}$ and $\{\varphi_k\}_{k=0}^{\infty}$ where $\lambda_0 = 0$, $\varphi_0(x) = 1$ and $\lambda_k = \left(\frac{k\pi}{L}\right)^2$ and $\varphi_k(x) = \cos \frac{k\pi x}{L}$ for $k = 1, 2, \ldots$. Let

$$f(x) = 2x^3 - 3Lx^2 \text{ for } 0 \leq x \leq L.$$

Find the series for f determined by the orthogonal sequence $\{\varphi_k\}_{k=0}^{\infty}$. Evaluate the integrals. The integration by parts formula is

$$\int_a^b u(x)v'(x)dx = [u(x)v(x)]_{x=a}^{x=b} - \int_a^b u'(x)v(x)dx$$

whenever each of u and v has a continuous derivative on $[a, b]$.

Solution. The series is $\{S_n\}$ where

$$S_n(x) = \sum_{k=0}^{n} \frac{< f, \varphi_k >}{< \varphi_k, \varphi_k >} \varphi_k(x).$$

$$< f, \varphi_0 >= \int_0^L (2x^3 - 3Lx^2) \cdot 1 dx = [\frac{1}{2}x^4 - Lx^3]_{x=0}^{x=L} = -\frac{1}{2}L^4$$

$$< \varphi_0, \varphi_0 >= \int_0^L 1 \cdot 1 dx = L$$

so

$$\frac{< f, \varphi_0 >}{< \varphi_0, \varphi_0 >} = -\frac{1}{2}L^3$$

For $k = 1, 2, \ldots$

$$\begin{aligned}
< \quad f, \varphi_k >&= \int_0^L (2x^3 - 3Lx^2) \cos \frac{k\pi x}{L} dx \\
&= \left[(2x^3 - 3Lx^2) \left(\frac{L}{k\pi} \right) \sin \frac{k\pi x}{L} \right]_{x=0}^{x=L} - \int_0^L (6x^2 - 6Lx) \left(\frac{L}{k\pi} \right) \sin \frac{k\pi x}{L} dx \\
&= \frac{6L}{k\pi} \int_0^L (Lx - x^2) \sin \frac{k\pi x}{L} dx \\
&= \frac{6L}{k\pi} \left(\left[(Lx - x^2)(\frac{-L}{k\pi}) \cos \frac{k\pi x}{L} \right]_{x=0}^{x=L} - \int_0^L (L - 2x)(\frac{-L}{k\pi}) \cos \frac{k\pi x}{L} dx \right) \\
&= \frac{6L^2}{(k\pi)^2} \int_0^L (L - 2x) \cos \frac{k\pi x}{L} dx \\
&= \frac{6L^2}{(k\pi)^2} \left(\left[(L - 2x)(\frac{L}{k\pi}) \sin \frac{k\pi x}{L} \right]_{x=0}^{x=L} - \int_0^L (-2)(\frac{L}{k\pi}) \sin \frac{k\pi x}{L} dx \right) \\
&= \frac{12L^3}{(k\pi)^3} \int_0^L \sin \frac{k\pi x}{L} dx = -\frac{12L^4}{(k\pi)^4} \left[\cos \frac{k\pi x}{L} \right]_{x=0}^{x=L} \\
&= -\frac{12L^4}{(k\pi)^4} (\cos k\pi - \cos 0) \\
&= \frac{12L^4}{(k\pi)^4} (1 - (-1)^k)
\end{aligned}$$

$$
\begin{aligned}
< \quad \varphi_k, \varphi_k >= & \int_0^L \cos^2 \frac{k\pi x}{L} dx \\
= & \int_0^L \frac{1}{2}(1+\cos\frac{2k\pi x}{L})dx \\
= & \left[\frac{1}{2}x + \frac{L}{2k\pi}\sin\frac{2k\pi x}{L}\right]_{x=0}^{x=L} \\
= & \frac{L}{2}
\end{aligned}
$$

Thus

$$
\frac{< f, \varphi_k >}{< \varphi_k, \varphi_k >} = \frac{2}{L}\frac{12L^4}{(k\pi)^4}(1-(-1)^k) = \frac{24L^3}{(k\pi)^4}(1-(-1)^k)
$$

Thus

$$
\begin{aligned}
S_n(x) &= \frac{< f, \varphi_0 >}{< \varphi_0, \varphi_0 >}\varphi_0(x) + \sum_{k=1}^{n} \frac{< f, \varphi_k >}{< \varphi_k, \varphi_k >}\varphi_k(x) \\
&= -\frac{1}{2}L^3 + \sum_{k=1}^{n} \frac{24L^3}{(k\pi)^4}(1-(-1)^k)\cos\frac{k\pi x}{L}
\end{aligned}
$$

So

$$
S_n(x) = -\frac{1}{2}L^3 + \frac{24L^3}{\pi^4}\sum_{k=1}^{n}\frac{(1-(-1)^k)}{k^4}\cos\frac{k\pi x}{L}
$$