Math 3363 Spring 2020 Homework 3

Due February 27

Please use a pencil and do the problems in the order in which they are listed. Write on only one side of each page and staple your pages.

You may use the following information without derivation.

- A proper listing of eigenvalues and eigenfunctions for

$$\begin{array}{ll} \text{(i)} & -\varphi''(x) = \lambda\varphi(x) \quad \text{for } 0 \le x \le L, \\ \text{(ii)} & \varphi(0) = 0, \text{ and} \\ \text{(iii)} & \varphi(L) = 0 \end{array}$$

  is $\{\lambda_k\}_{k=1}^{\infty}$ and $\{\varphi_k\}_{k=1}^{\infty}$ where $\lambda_k = (\frac{k\pi}{L})^2$ and $\varphi_k(x) = \sin\frac{k\pi x}{L}$.

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  is $\{\lambda_k\}_{k=0}^{\infty}$ and $\{\varphi_k\}_{k=0}^{\infty}$ where $\lambda_k = (\frac{k\pi}{L})^2$ and $\varphi_k(x) = \cos\frac{k\pi x}{L}$. Note that $\lambda_0 = 0$ and $\varphi_0(x) = 1$.

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  is $\{\lambda_k\}_{k=1}^{\infty}$ and $\{\varphi_k\}_{k=1}^{\infty}$ where $\lambda_k = \left(\frac{(2k-1)\pi}{2L}\right)^2$ and $\varphi_k(x) = \sin\frac{(2k-1)\pi x}{2L}$.

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1. Find a function v of the form

$$v(x, y) = ax + by + cxy + d$$

such that $v(0, 0) = 2$, $v(8, 0) = -1$, $v(8, 4) = 3$, and $v(0, 4) = -4$.

2. Let v be as in Problem 1. Show that

$$\frac{\partial^2 v}{\partial x^2}(x, y) + \frac{\partial^2 v}{\partial y^2}(x, y) = 0$$

for all (x, y) in the plane.

3. Consider the following problem for Laplace's equation in a rectangle.

$$\frac{\partial^2 u}{\partial x^2}(x, y) + \frac{\partial^2 u}{\partial y^2}(x, y) = 0$$

for all (x, y) in the rectangle $[0, 8] \times [0, 4]$ which is all (x, y) where $0 \leq x \leq 8$ and $0 \leq y \leq 4$ and

$$u(x, y) = F(x, y)$$

for all (x, y) on the boundary of $[0, 8] \times [0, 4]$ where

$$F(x, y) = x^3 + y^2 + xy + 2$$

Find the function v of the form

$$v(x, y) = ax + by + cxy + d$$

such that

$$v(x, y) = F(x, y)$$

at each of the four corners of the rectangle. (This is not the same v as in Problem 1.) Then let w be given by

$$w(x, y) = u(x, y) - v(x, y)$$

for all (x, y) in the rectangle $[0, 8] \times [0, 4]$. Complete but do not solve the following problem statement for w.

$$\frac{\partial^2 w}{\partial x^2}(x, y) + \frac{\partial^2 w}{\partial y^2}(x, y) = ?$$

for all (x, y) in the rectangle $[0, 8] \times [0, 4]$,

$$w(x, 0) = ?$$

for $0 \leq x \leq 8$,

$$w(x, 2) = ?$$

for $0 \leq x \leq 8$,

$$w(0, y) = ?$$

for $0 \leq y \leq 4$, and

$$w(4, y) = ?$$

for $0 \leq y \leq 4$. Check that $w(x, y) = 0$ at each of the four corners of the rectangle.

4. Derive the solution to

$$
\begin{aligned}
\frac{\partial^2 u}{\partial x^2}(x,y) + \frac{\partial^2 u}{\partial y^2}(x,y) &= 0, && (1)\\
\frac{\partial u}{\partial x}(0,y) &= 0, && (2)\\
u(L,y) &= 0, && (3)\\
\frac{\partial u}{\partial y}(x,0) &= 0, \text{ and} && (4)\\
u(x,0) &= f(x) && (5)
\end{aligned}
$$

for $0 \le x \le L$ and $0 \le y \le H$. Derive the solution. Don't just give the final answer. Some similar problems, some of which have answers, can be found on page 81 of the text.