Math 3363 Spring 2020 Homework 3

Solutions

Please use a pencil and do the problems in the order in which they are listed. Write on only one side of each page and staple your pages.

You may use the following information without derivation.

- A proper listing of eigenvalues and eigenfunctions for

$$\begin{array}{ll} \text{(i)} & -\varphi''(x) = \lambda\varphi(x) \quad \text{for } 0 \le x \le L, \\ \text{(ii)} & \varphi(0) = 0, \text{ and} \\ \text{(iii)} & \varphi(L) = 0 \end{array}$$

  is $\{\lambda_k\}_{k=1}^{\infty}$ and $\{\varphi_k\}_{k=1}^{\infty}$ where $\lambda_k = (\frac{k\pi}{L})^2$ and $\varphi_k(x) = \sin\frac{k\pi x}{L}$.

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  is $\{\lambda_k\}_{k=0}^{\infty}$ and $\{\varphi_k\}_{k=0}^{\infty}$ where $\lambda_k = (\frac{k\pi}{L})^2$ and $\varphi_k(x) = \cos\frac{k\pi x}{L}$. Note that $\lambda_0 = 0$ and $\varphi_0(x) = 1$.

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  is $\{\lambda_k\}_{k=1}^{\infty}$ and $\{\varphi_k\}_{k=1}^{\infty}$ where $\lambda_k = \left(\frac{(2k-1)\pi}{2L}\right)^2$ and $\varphi_k(x) = \sin\frac{(2k-1)\pi x}{2L}$.

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  is $\{\lambda_k\}_{k=1}^{\infty}$ and $\{\varphi_k\}_{k=1}^{\infty}$ where $\lambda_k = \left(\frac{(2k-1)\pi}{2L}\right)^2$ and $\varphi_k(x) = \cos\frac{(2k-1)\pi x}{2L}$.

1. Find a function v of the form

$$v(x, y) = ax + by + cxy + d$$

such that $v(0, 0) = 2$, $v(8, 0) = -1$, $v(8, 4) = 3$, and $v(0, 4) = -4$.

Solution.

$$\begin{aligned}
d &= 2 \\
8a + 2 &= -1 \\
a &= -\frac{3}{8} \\
4b + 2 &= -4 \\
b &= -\frac{3}{2} \\
-3 + -6 + 32c + 2 &= 3 \\
c &= \frac{5}{16} \\
v(x, y) &= -\frac{3}{8}x - \frac{3}{2}y + \frac{5}{16}xy + 2
\end{aligned}$$

2. Let v be as in Problem 1. Show that

$$\frac{\partial^2 v}{\partial x^2}(x, y) + \frac{\partial^2 v}{\partial y^2}(x, y) = 0$$

for all (x, y) in the plane.

Solution.

$$\begin{aligned}
\frac{\partial v}{\partial x}(x, y) &= -\frac{3}{8} + \frac{5}{16}y \\
\frac{\partial^2 v}{\partial x^2}(x, y) &= 0 \\
\frac{\partial v}{\partial y}(x, y) &= -\frac{3}{2} + \frac{5}{16}x \\
\frac{\partial^2 v}{\partial y^2}(x, y) &= 0
\end{aligned}$$

so

$$\frac{\partial^2 v}{\partial x^2}(x, y) + \frac{\partial^2 v}{\partial y^2}(x, y) = 0$$

3. Consider the following problem for Laplace's equation in a rectangle.

$$\frac{\partial^2 u}{\partial x^2}(x,y) + \frac{\partial^2 u}{\partial y^2}(x,y) = 0$$

for all (x,y) in the rectangle $[0,8] \times [0,4]$ which is all (x,y) where $0 \le x \le 8$ and $0 \le y \le 4$ and

$$u(x,y) = F(x,y)$$

for all (x,y) on the boundary of $[0,8] \times [0,4]$ where

$$F(x,y) = x^3 + y^2 + xy + 2$$

Find the function v of the form

$$v(x,y) = ax + by + cxy + d$$

such that

$$v(x,y) = F(x,y)$$

at each of the four corners of the rectangle. (This is not the same v as in Problem 1.) Then let w be given by

$$w(x,y) = u(x,y) - v(x,y)$$

for all (x,y) in the rectangle $[0,8] \times [0,4]$. Complete but do not solve the following problem statement for w.

$$\frac{\partial^2 w}{\partial x^2}(x,y) + \frac{\partial^2 w}{\partial y^2}(x,y) = ?$$

for all (x,y) in the rectangle $[0,8] \times [0,4]$,

$$w(x,0) = ?$$

for $0 \le x \le 8$,

$$w(x,2) = ?$$

for $0 \le x \le 8$,

$$w(0,y) = ?$$

for $0 \le y \le 4$, and

$$w(4,y) = ?$$

for $0 \le y \le 4$. Check that $w(x,y) = 0$ at each of the four corners of the rectangle.

Solution.

$$d = v(0,0) = F(0,0) = 2$$

$$8a + d = v(8,0) = F(8,0) = 514$$

$$8a = 512$$

$$a = 64$$

$$4b + d = v(0,4) = F(0,4) = 18$$

$$4b = 16$$

$$b = 4$$

$$64 \cdot 8 + 4 \cdot 4 + 32c + 2 = v(8,4) = F(8,4) = 562$$

$$64 \cdot 8 + 4 \cdot 4 + 32c + 2 = 562$$

$$c = 1$$

$$v(x,y) = 64x + 4y + xy + 2$$

$$\begin{aligned} \frac{\partial^2 w}{\partial x^2}(x,y) + \frac{\partial^2 w}{\partial y^2}(x,y) &= (\frac{\partial^2 u}{\partial x^2}(x,y) + \frac{\partial^2 u}{\partial y^2}(x,y)) - (\frac{\partial^2 v}{\partial x^2}(x,y) + \frac{\partial^2 v}{\partial y^2}(x,y)) \\ &= 0 - 0 = 0 \end{aligned}$$

$$\begin{aligned} w(x,0) &= u(x,0) - v(x,0) = F(x,0) - v(x,0) \\ &= x^3 - 64x \end{aligned}$$

$$\begin{aligned} w(x,2) &= u(x,2) - v(x,2) = F(x,2) - v(x,2) \\ &= x^3 - 64x - 4 \end{aligned}$$

$$\begin{aligned} w(0,y) &= u(0,y) - v(0,y) = F(0,y) - v(0,y) \\ &= y^2 - 4y \end{aligned}$$

$$\begin{aligned} w(4,y) &= u(4,y) - v(4,y) = F(4,y) - v(4,y) \\ &= y^2 - 4y - 192 \end{aligned}$$

$$w(0,0) = F(0,0) - v(0,0) = 2 - 2 = 0$$

$$w(8,0) = F(8,0) - v(8,0) = 514 = 514 = 0$$

$$w(8,4) = F(8,4) - v(8,4) = 562 - 562 = 0$$

$$w(0,4) = F(0,4) - v(0,4) = 18 - 18 = 0$$

4. Derive the solution to

$$\begin{aligned}
\frac{\partial^2 u}{\partial x^2}(x,y) + \frac{\partial^2 u}{\partial y^2}(x,y) &= 0, && (1)\\
\frac{\partial u}{\partial x}(0,y) &= 0, && (2)\\
u(L,y) &= 0, && (3)\\
\frac{\partial u}{\partial y}(x,0) &= 0, \text{ and} && (4)\\
u(x,0) &= f(x) && (5)
\end{aligned}$$

for $0 \le x \le L$ and $0 \le y \le H$. Derive the solution. Don't just give the final answer. Some similar problems, some of which have answers, can be found on page 81 of the text.

Solution. Suppose that u is an elementary separated solution of (1) with

$$u(x,y) = \varphi(x)h(y). \tag{6}$$

Then

$$\varphi''(x)h(y) + \varphi(x)h''(y) = 0 \tag{7}$$

and that

$$\varphi(x)h(y) \neq 0$$

for all (x,y) with $0 \le x \le L$ and $0 \le y \le H$. Dividing each side of (7) by $\varphi(x)h(y)$ produces

$$\frac{\varphi''(x)}{\varphi(x)} + \frac{h''(y)}{h(y)} = 0$$

or

$$\frac{h''(y)}{h(y)} = -\frac{\varphi''(x)}{\varphi(x)}.$$

Since this is true for $0 \le x \le L$ and $0 \le y \le H$ there is a constant λ such that

$$\frac{h''(y)}{h(y)} = \lambda = -\frac{\varphi''(x)}{\varphi(x)}$$

For $0 \le x \le L$ and $0 \le y \le H$. From this, it follows that

$$-\varphi''(x) = \lambda\varphi(x) \text{ for } 0 \le x \le L \tag{8}$$

and

$$h''(y) - \lambda h(y) = 0 \text{ for } 0 \le y \le H. \tag{9}$$

Starting over, if (6), (8), and (9) hold then

$$\frac{\partial^2 u}{\partial x^2}(x,y) = \varphi''(x)h(y) = -\lambda\varphi(x)h(y)$$

and

$$\frac{\partial^2 u}{\partial y^2}(x,y) = \varphi(x)h''(y) = \lambda\varphi(x)h(y)$$

so (1) holds without assuming $\varphi(x)h(y) \neq 0$ for all (x,y) with $0 \leq x \leq L$ and $0 \leq y \leq H$. In view of (5), we need

$$\varphi(x_0)h(y_0) \neq 0 \tag{10}$$

for some (x_0, y_0) in $[0,L] \times [0,H]$. In view of (10), if (2), (3), and (4) hold, then

$$\varphi'(0) = 0, \tag{11}$$

$$\varphi(L) = 0, \tag{12}$$

and

$$h'(0) = 0. \tag{13}$$

A proper listing of eigenvalues and eigenfunctions for (8), (11), and (12) is $\{\lambda_k\}_{k=1}^{\infty}$ and $\{\varphi_k\}_{k=1}^{\infty}$ where $\lambda_k = \left(\frac{(2k-1)\pi}{2L}\right)^2$ and $\varphi_k(x) = \cos\frac{(2k-1)\pi x}{2L}$.

A there are a number of linearly independent pairs of solutions to (9). For this problem, the best way to describe the solutions to (9) is as follows. h is a solution to (9) if and only if

$$h(y) = c_1 \cosh\sqrt{\lambda}y + c_2 \cosh\sqrt{\lambda}(H-y)$$

in which case

$$h'(y) = c_1\sqrt{\lambda}\sinh\sqrt{\lambda}y - c_2\sqrt{\lambda}\sinh\sqrt{\lambda}(H-y).$$

Thus when $\lambda = \lambda_k$, the solutions to (9) and (13) are multiples of h_k where

$$h_k(y) = \cosh\sqrt{\lambda_k}y.$$

For each k we have an elementary separated solution u_k to (1)-(4) given by

$$u_k(x,y) = \varphi_k(x)h_k(y).$$

Since (1)-(4) is linear and homogeneous, anything of the form

$$\sum_{k=1}^{n} E_k u_k(x,y)$$

will also be a solution. We conjecture that the solution u to (1)-(5) is of the form

$$u(x,y) = \sum_{k=1}^{\infty} E_k u_k(x,y).$$

Condition (5) will hold if and only if

$$f(x) = \sum_{k=1}^{\infty} E_k u_k(x,0) = \sum_{k=1}^{\infty} E_k \varphi_k(x)h_k(0),$$

and this will hold if and only if

$$E_k h_k(0) = \frac{< f, \varphi_k >}{< \varphi_k, \varphi_k >}.$$

$$h_k(0) = 1$$

We expect that the solution is given by

$$u(x, y) = \sum_{k=1}^{\infty} E_k \cos \sqrt{\lambda_k} x \cosh \sqrt{\lambda_k} y$$

where

$$E_k = \frac{2}{L} \int_0^L f(x) \cos \sqrt{\lambda_k} x dx$$

in which

$$\lambda_k = \left(\frac{(2k-1)\pi}{2L} \right)^2$$

so that

$$\sqrt{\lambda_k} = \frac{(2k-1)\pi}{2L}.$$