Math 3363 Spring 2020 Homework 4

Solutions

Do the problems in the order in which they are listed. Submit your work as a pdf file.

1. Let f be given by
$$f(x) = (x+L)^2 \text{ for } -L \leq x \leq L.$$
Find the Fourier series for f. Recall that the series is $\{S_n\}_{n=1}^{\infty}$ where
$$S_n(x) = A_0 + \sum_{k=1}^{n}[A_k \cos\frac{k\pi x}{L} + B_k \sin\frac{k\pi x}{L}].$$

Solution.
$$A_0 = \frac{1}{2L}\int_{-L}^{L}(x+L)^2 dx = \frac{4}{3}L^2$$

$$\begin{aligned} A_k &= \frac{1}{L}\int_{-L}^{L}(x+L)^2\cos\frac{k\pi x}{L}dx \\ & \quad \text{integrate by parts twice} \\ &= \frac{4L^2}{\pi^2}\frac{(-1)^k}{k^2} \end{aligned}$$

$$\begin{aligned} B_k &= \frac{1}{L}\int_{-L}^{L}(x+L)^2\sin\frac{k\pi x}{L}dx \\ & \quad \text{integrate by parts twice} \\ &= -\frac{4L^2}{\pi}\frac{(-1)^k}{k} \end{aligned}$$

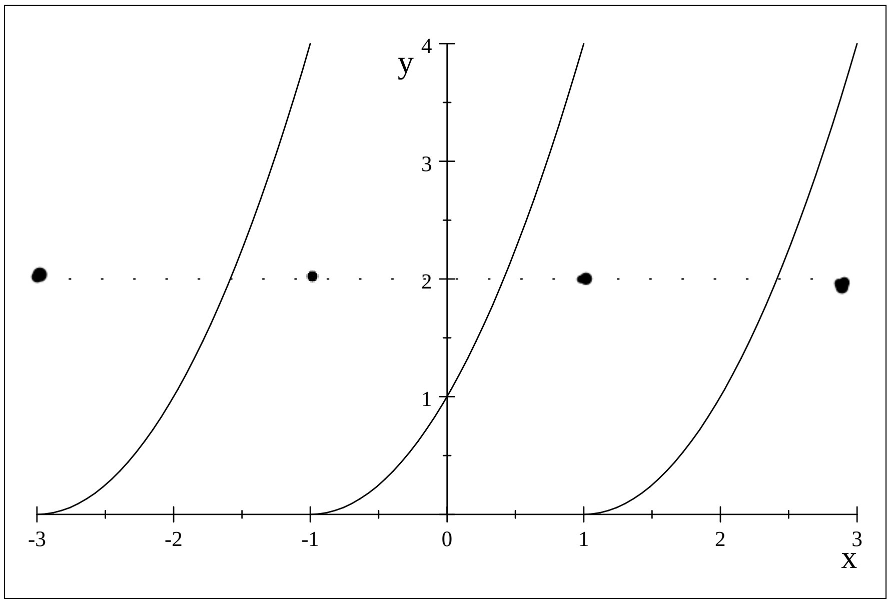
The Fourier Series is $\{S_n\}$ where

$$\begin{aligned} S_n(x) &= \frac{4}{3}L^2 + \sum_{k=1}^{n}[\frac{4L^2}{\pi^2}\frac{(-1)^k}{k^2}\cos\frac{k\pi x}{L} - \frac{4L^2}{\pi}\frac{(-1)^k}{k}\sin\frac{k\pi x}{L}] \\ &= \frac{4}{3}L^2 + \frac{4L^2}{\pi^2}\sum_{k=1}^{n}\frac{(-1)^k}{k^2}[\cos\frac{k\pi x}{L} - k\pi\sin\frac{k\pi x}{L}] \end{aligned}$$

2. Let f be as in Problem 1 and let $L = 1$. Sketch the graph of h over $[-3, 3]$ where h is the limit of the Fourier series. Be sure to show the value of h at each number in $[-3, 3]$. Recall that

$$h(x) = A_0 + \sum_{k=1}^{\infty} [A_k \cos k\pi x + B_k \sin k\pi x].$$

Solution. The value of h is 2 at each odd integer. Otherwise, h is the period 2 extension of the restriction of f to the open interval $(-1, 1)$.



h

3. Let f be given by

$$f(x) = x^2 \text{ for } 0 \le x \le L.$$

Find the sine series for f. Recall that the series is $\{S_n\}_{n=1}^{\infty}$ where

$$S_n(x) = \sum_{k=1}^{n} B_k \sin \frac{k\pi x}{L}.$$

Solution. Integration by parts twice shows that

$$B_k = \frac{2}{L} \int_0^L x^2 \sin \frac{k\pi x}{L} dx = 2L^2 \left[\frac{(-1)^{k+1}}{k\pi} + \frac{2((-1)^k - 1)}{k^3\pi^3} \right]$$

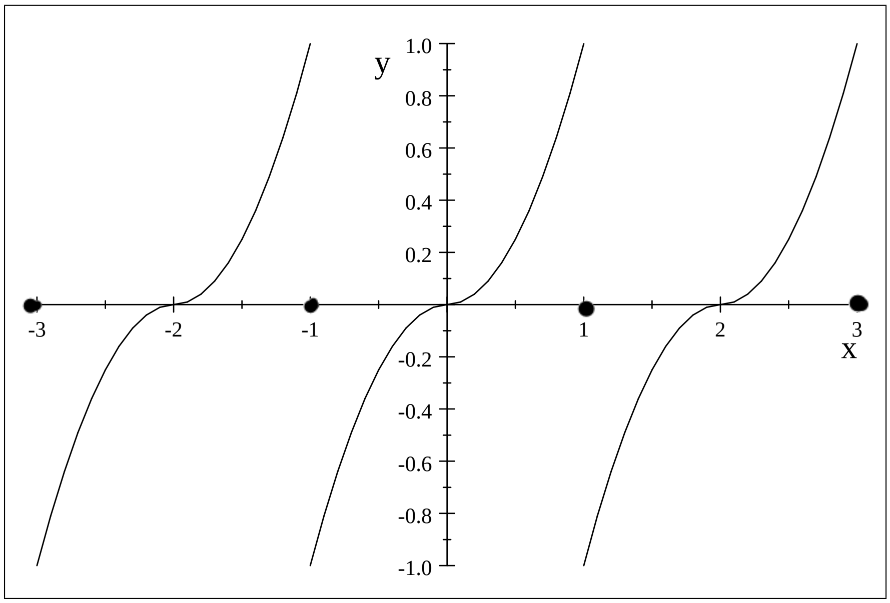
So the sine series is $\{S_n\}$ where

$$S_n(x) = 2L^2 \sum_{k=1}^{n} \left[\frac{(-1)^{k+1}}{k\pi} + \frac{2((-1)^k - 1)}{k^3\pi^3} \right] \sin \frac{k\pi x}{L}.$$

4. Let f be as in Problem 3 and let $L = 1$. Sketch the graph of h over $[-3, 3]$ where h is the limit of the sine series. Be sure to show the value of h at each number in $[-3, 3]$. Recall that

$$h(x) = \sum_{k=1}^{\infty} B_k \sin \frac{k\pi x}{L}.$$

Solution. The value of h is 0 at each odd integer. Otherwise, h is the period 2 extension of the odd $(-1, 1)$ extension of f.



h

5. Let f be given by

$$f(x) = \frac{L}{2} - x \text{ for } 0 \leq x \leq L.$$

Find the cosine series for f. Recall that the series is $\{S_n\}_{n=1}^{\infty}$ where

$$S_n(x) = A_0 + \sum_{k=1}^{n} A_k \cos \frac{k\pi x}{L}.$$

Solution.

$$A_0 = \frac{1}{L} \int_0^L (\frac{L}{2} - x)dx = 0$$

Integration by parts shows that

$$A_k = \frac{2}{L} \int_0^L (\frac{L}{2} - x) \cos \frac{k\pi x}{L} dx = \frac{2L(1 - (-1)^k)}{\pi^2 k^2}$$

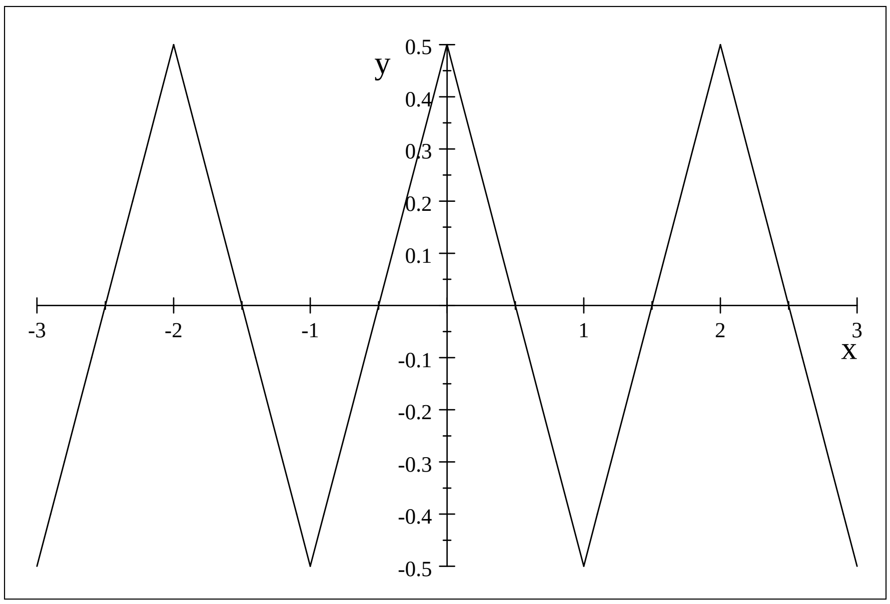
The cosine series is $\{S_n\}$ where

$$S_n(x) = \frac{2L}{\pi^2} \sum_{k=1}^{n} \frac{(1-(-1)^k)}{k^2} \cos \frac{k\pi x}{L}$$

6. Let f be as in Problem 5 and let $L = 1$. Sketch the graph of h over $[-3, 3]$ where h is the limit of the cosine series. Be sure to show the value of h at each number in $[-3, 3]$. Recall that

$$h(x) = A_0 + \sum_{k=1}^{\infty} A_k \cos \frac{k\pi x}{L}.$$

Solution. h is the period 2 extension of the even $[-1, 1]$extension of f.



h

7. Let f be given by

$$f(x) = (x-1)^3$$

for all numbers x. Find and simplify the even part of f and the odd part of f.

Solution.

$$f_e(x) = \frac{1}{2}(f(x) + f(-x)) = \frac{1}{2}((x-1)^3 + (-x-1)^3) = -3x^2 - 1$$

and

$$f_o(x) = \frac{1}{2}(f(x) - f(-x)) = \frac{1}{2}((x-1)^3 - (-x-1)^3) = x^3 + 3x$$

Second way:

$$f(x) = (x-1)^3 = x^3 - 3x^2 + 3x - 1 = (-3x^2 - 1) + (x^3 + 3x)$$

so

$$f_e(x) = -3x^2 - 1$$

and

$$f_o(x) = x^3 + 3x$$

8. Let f be given by

$$f(x) = e^x$$

for all numbers x. Find the even part of f and the odd part of f. Express each part in terms of an elementary function.

Solution.

$$f_e(x) = \frac{1}{2}(f(x) + f(-x)) = \frac{1}{2}(e^x + e^{-x}) = \cosh x$$

and

$$f_o(x) = \frac{1}{2}(f(x) - f(-x)) = \frac{1}{2}(e^x + e^{-x}) = \sinh x$$