Math 3364 Spring 2020 Homework 5 Solutions

Due April 16 at 11:59 p.m.

Do the problems in the order in which they are listed. Submit your work in a pdf file.

1. Derive the solution to the following problem for Laplace's equation in a rectangular solid.

$$\begin{aligned}
\frac{\partial^2 u}{\partial x^2}(x,y,z) + \frac{\partial^2 u}{\partial y^2}(x,y,z) + \frac{\partial^2 u}{\partial z^2}(x,y,z) &= 0 && (1)\\
u(0,y,z) &= 0 && (2)\\
u(L,y,z) &= 0 && (3)\\
u(x,0,z) &= 0 && (4)\\
u(x,H,z) &= 0 && (5)\\
u(x,y,0) &= 0 \text{ and} && (6)\\
u(x,y,T) &= f(x,y) && (7)
\end{aligned}$$

for $0 \le x \le L$, $0 \le y \le H$, and $0 \le z \le T$. Start by looking for elementary separated solutions of the form

$$u(x,y,z) = \varphi(x,y)h(z). \qquad (8)$$

Use the eigenvalues and eigenfunction found in the note, 'A two-dimensional eigenvalue problem,' found on Dr. Walker's web site.

Solution Outline. Starting with (8), separation of variables produces

$$-\left(\frac{\partial^2 \varphi}{\partial x^2}(x,y) + \frac{\partial^2 \varphi}{\partial y^2}(x,y)\right) = \lambda\varphi(x,y) \text{ for } (x,y) \text{ in } [0,L]\times[0,H] \qquad (9)$$

and

$$h'(z) - \lambda h(z) = 0 \text{ for } 0 \le z \le T. \qquad (10)$$

The conditions (2)-(6) produce

$$\varphi(x,y) = 0 \text{ for } (x,y) \text{ on the boundary of } [0,L]\times[0,H] \qquad (11)$$

and

$$h(0) = 0. \qquad (12)$$

A proper listing of eigenvalues and eigenfunctions for (9) and (11) is $\{\lambda_{kj}\}_{k,j=1}^{\infty}$ and $\{\varphi_{kj}\}_{k,j=1}^{\infty}$ where $\lambda_{kj} = (\frac{k\pi}{L})^2 + (\frac{j\pi}{H})^2$ and $\varphi_{kj}(x,y) = \sin\frac{k\pi x}{L}\sin\frac{j\pi y}{H}$. When $\lambda = \lambda_{kj}$, the solutions to (10) and (12) are multiples of h_{kj} where $h_{kj}(z) = \sinh\sqrt{\lambda_{kj}}z$. Since the problem consisting of (1)-(6) is linear and homogeneous, anything of the form

$$\sum_{k=1}^{n}\sum_{j=1}^{m} E_{kj}\varphi_{kj}(x,y)h_{kj}(z)$$

will be a solution. In order to have (7) also satisfied, an infinite sum will be needed. We conjecture that

$$u(x,y,z) = \sum_{k=1}^{\infty}\sum_{j=1}^{\infty} E_{kj}\varphi_{kj}(x,y)h_{kj}(z).$$

Condition (7) requires

$$f(x,y) = \sum_{k=1}^{\infty}\sum_{j=1}^{\infty} E_{kj}\varphi_{kj}(x,y)h_{kj}(T)$$

so

$$E_{kj} = \frac{4}{LH\sinh\sqrt{\lambda_{kj}}T}\int_0^L\int_0^H f(x,y)\varphi_{kj}(x,y)dydx$$

The solution is given by

$$u(x,y,z) = \sum_{k=1}^{\infty}\sum_{j=1}^{\infty} E_{kj}\sin\frac{k\pi x}{L}\sin\frac{j\pi y}{H}\sinh\sqrt{(\frac{k\pi}{L})^2 + (\frac{j\pi}{H})^2}z$$

where

$$E_{kj} = \frac{4}{LH\sinh\sqrt{(\frac{k\pi}{L})^2 + (\frac{j\pi}{H})^2}T}\int_0^L\int_0^H f(x,y)\sin\frac{k\pi x}{L}\sin\frac{j\pi y}{H}dydx$$

2. Find a proper listing of eigenvalues and eigenfunctions for the following problem

$$\begin{aligned}
-\left(\frac{\partial^2\varphi}{\partial x^2}(x,y) + \frac{\partial^2\varphi}{\partial y^2}(x,y)\right) &= \lambda\varphi(x,y) \text{ for } 0 \le x \le L \text{ and } 0 \le y \le H, && (1)\\
\frac{\partial\varphi}{\partial x}(0,y) &= 0 \text{ for } 0 \le y \le H, && (2)\\
\varphi(L,y) &= 0 \text{ for } 0 \le y \le H, && (3)\\
\varphi(x,0) &= 0 \text{ for } 0 \le x \le L, \text{ and} && (4)\\
\varphi(x,H) &= 0 \text{ for } 0 \le x \le L. && (5)
\end{aligned}$$

Note that zero is an eigenvalue of one of the related one-dimensional problems.

Solution. Actually, zero is not an eigenvalue of one of the related one-dimensional problems. I was thinking of the problem where $\frac{\partial\varphi}{\partial x}(L,y)=0$ rather than $\varphi(L,y)=0$ as is the case in this problem.

We begin by looking for an elementary separated solution of the form

$$\varphi(x,y)=f(x)g(y) \tag{6}$$

Putting this form into (1) we have

$$-\left(f''(x)g(y)+f(x)g''(y)\right)=\lambda f(x)g(y) \text{ for } 0\le x\le L \text{ and } 0\le y\le H.$$

Dividing each side of this equation by $f(x)g(y)$ (assuming for now that this is not zero) we have

$$-\frac{f''(x)}{f(x)}=\lambda+\frac{g''(y)}{g(y)} \text{ for } 0\le x\le L \text{ and } 0\le y\le H.$$

Letting μ be the common constant value we have

$$-f''(x)=\mu f(x) \text{ for } 0\le x\le L \tag{7}$$

and

$$g''(y)+\lambda g(y)=\mu g(y)$$

or

$$-g''(y)=(\lambda-\mu)g(y)$$

or

$$-g''(y)=\delta g(y) \text{ for } 0\le y\le H \tag{8}$$

where

$$\delta=\lambda-\mu. \tag{9}$$

Note that

$$\lambda=\mu+\delta. \tag{10}$$

From (6), (2), and (3) we have

$$f'(0)=0 \tag{11}$$

and

$$f(L)=0. \tag{12}$$

From (6), (4), and (5) we have

$$g(0)=0 \tag{13}$$

and

$$g(H)=0. \tag{14}$$

The problem (7), (11), (12) and the problem (8), (13), (14) are copies of problems that were studied in Section2.1. Proper listings of eigenvalues and eigenfunctions are

$$\{\mu_k\}_{k=1}^{\infty} \text{ and } \{f_k\}_{k=1}^{\infty}$$

and

$$\{\delta_j\}_{j=1}^{\infty} \text{ and } \{g_j\}_{j=1}^{\infty}$$

where

$$\mu_k = (\frac{(2k-1)\pi}{2L})^2,$$

$$f_k(x) = \cos\frac{(2k-1)\pi x}{2L},$$

$$\delta_j = (\frac{j\pi}{H})^2,$$

and

$$g_j(y) = \sin\frac{j\pi y}{H}$$

for $0 \le x \le L$, $0 \le y \le H$, $k = 1, 2, \ldots$, and $j = 1, 2, \ldots$. Note that we have used different (k and j) index symbols for the two listings.

In view of (6) and (10), a proper listing of eigenvalues and eigenfunctions for (1)-(5) is given by

$$\{\lambda_{kj}\}_{k=1,j=1}^{\infty} \text{ and } \{\varphi_{kj}\}_{k=1,j=1}^{\infty}$$

where

$$\lambda_{kj} = \mu_k + \delta_j = (\frac{(2k-1)\pi}{2L})^2 + (\frac{j\pi}{H})^2$$

and

$$\varphi_{kj}(x,y) = \cos\frac{(2k-1)\pi x}{2L}\sin\frac{j\pi y}{H}$$

for $0 \le x \le L$, $0 \le y \le H$, $k = 1, 2, \ldots$, and $j = 1, 2, \ldots$.

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3. Derive the solution for the following heat equation problem for a rectangular plate.

$$\begin{aligned}
\frac{\partial u}{\partial t}(x,y,t) &= \kappa\left(\frac{\partial^2 u}{\partial x^2}(x,y,t) + \frac{\partial^2 u}{\partial y^2}(x,y,t)\right) && (1)\\
\frac{\partial u}{\partial x}(0,y,t) &= 0 && (2)\\
u(L,y,t) &= 0 && (3)\\
u(x,0,t) &= 0 && (4)\\
u(x,H,t) &= 0 \text{ and} && (5)\\
u(x,y,0) &= \alpha(x,y) && (6)
\end{aligned}$$

for $0 \le x \le L$, $0 \le y \le H$, and $t \ge 0$. Use the eigenvalues and eigenfunctions that you found in Problem 2.

Solution. Starting with

$$u(x,y,t) = \varphi(x,y)h(t), \tag{7}$$

separation of variables produces

$$-\left(\frac{\partial^2\varphi}{\partial x^2}(x,y) + \frac{\partial^2\varphi}{\partial y^2}(x,y)\right) = \lambda\varphi(x,y) \tag{8}$$

for (x,y) in $[0,L]\times[0,H]$ and

$$h'(t) + \lambda h(t) = 0 \tag{9}$$

for $t \geq 0$. Assuming that u is not the zero function, conditions (2)-(5) imply

$$\begin{aligned}
\frac{\partial\varphi}{\partial x}(0,y) &= 0 \text{ for } 0 \leq y \leq H, &(10)\\
\varphi(L,y) &= 0 \text{ for } 0 \leq y \leq H, &(11)\\
\varphi(x,0) &= 0 \text{ for } 0 \leq x \leq L, \text{ and} &(12)\\
\varphi(x,H) &= 0 \text{ for } 0 \leq x \leq L. &(13)
\end{aligned}$$

A proper listing of eigenvalues and eigenfunctions for (8) and (10)-(13) is $\{\lambda_{kj}\}_{k=1.j=1}^{\infty}$ and $\{\varphi_{kj}\}_{k=1,,j=1}^{\infty}$as given in Problem 2.

When $\lambda = \lambda_{kj}$,the solutions to (9) are multiples of h_{kj} where

$$h_{kj}(t) = e^{-\lambda_{kj}t}$$

Any linear combination of the functions $\varphi_{kj}h_{kj}$will be a solution to (1)-(5). To get a solution to (6) also, we will need an infinite sum.

$$u(x,y,t) = \sum_{k=1}^{\infty}\sum_{j=1}^{\infty} E_{kj}\varphi_{kj}(x,y)h_{kj}(t).$$

Noting that

$$h_{kj}(0) = 1$$

we see that (6) will hold if and only if

$$\alpha(x,y) = \sum_{k=1}^{\infty}\sum_{j=1}^{\infty} E_{kj}\varphi_{kj}(x,y).$$

This will hold if and only if

$$E_{kj} = \frac{<\alpha,\varphi_{kj}>}{<\varphi_{kj},\varphi_{kj}>}$$

Thus we conjecture that the solution to (1)-(6) is given by

$$u(x,y,t) = \sum_{k=1}^{\infty}\sum_{j=1}^{\infty} E_{kj}\cos\frac{(2k-1)\pi x}{2L}\sin\frac{j\pi y}{H}\exp(-((\frac{(2k-1)\pi}{2L})^2 + (\frac{j\pi}{H})^2)t)$$

where

$$E_{kj} = \frac{4}{LH}\int_0^L\int_0^H \alpha(x,y)\cos\frac{(2k-1)\pi x}{2L}\sin\frac{k\pi y}{H}dydx$$

for $k = 1,2,\ldots$ and $j = 1,2,\ldots$.