Math 3363 Examination I Solutions

Spring 2020

Please use a pencil and do the problems in the order in which they are listed. No books, notes, calculators, cell phones, smart watches, or other electronics.

1. A rod of length L (units of length), insulated except perhaps at its ends, lies along the x-axis with its left end at coordinate 0 and its right end at coordinate L. Let e, ϕ, and Q be as follows. The thermal energy density (energy/length) at t (units of time after the time origin) at points with first coordinate x is $e(x,t)$. The heat flux (energy/time) to the right at time t through the cross section consisting of points with first coordinate x is $\phi(x,t)$. (A negative value for $\phi(x,t)$ indicates heat flow to the left.) The heat energy per unit length being generated per unit time inside the rod at time t at points with first coordinate x is $Q(x,t)$. (A negative value for Q indicates a heat sink.) Suppose that Q is continuous and that e and ϕ have continuous partial derivatives.

   (a) Suppose that $0 \leq a \leq b \leq L$. What is the total thermal energy in the segment from a to b at time t?

   .**Solution.**

   $$\int_a^b e(x,t)dx$$

   (b) One way to express the rate of change of this thermal energy is

   $$\phi(a,t) - \phi(b,t) + \int_a^b Q(x,t)dx.$$

   Express this quantity as a single integral.

   Solution.

   $$\int_a^b \left(-\frac{\partial \phi}{\partial x}(x,t) + Q(x,t)\right) dx$$

(c) Starting with your expression in Part (a) for the total energy in the segment from a to b, give a second way to express the rate of change of this thermal energy as an integral.

Solution.

$$\frac{d}{dt}\int_a^b e(x,t)dx = \int_a^c \frac{\partial e}{\partial t}(x,t)dx$$

(d) In addition to the information given in the statement of the problem, let $c(x)$ be the specific heat, $K_0(x)$ be the thermal conductivity, and $\rho(x)$ be the mass density at points with first coordinate x, and let $u(x,t)$ be the temperature at time t at points with first coordinate x. Starting with the equation

$$\frac{\partial e}{\partial t} = -\frac{\partial \phi}{\partial x} + Q \text{ for } 0 \leq x \leq L \text{ and } t \geq 0.$$

Derive the equation

$$c\rho\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}(K_0\frac{\partial u}{\partial x}) + Q \text{ for } 0 \leq x \leq L \text{ and } t \geq 0.$$

Solution. According to the definition of temperature,

$$e = c\rho(u - Z)$$

where Z is a constant. So

$$c\rho\frac{\partial u}{\partial t} = -\frac{\partial \phi}{\partial x} + Q.$$

According to Fourier's law of heat conduction

$$\phi = -K_0\frac{\partial u}{\partial x,}$$

so

$$c\rho\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}(K_0\frac{\partial u}{\partial x}) + Q$$

2. Find the constant β so that the following problem has an equilibrium solution.

$$\frac{\partial w}{\partial t}(x,t) = \frac{\partial^2 w}{\partial x^2}(x,t) + 1 \text{ for } t \geq 0 \text{ and } 0 \leq x \leq 5,$$

$$w(x,0) = x \text{ for } 0 \leq x \leq 5,$$

$$\frac{\partial w}{\partial x}(0,t) = 1, \text{ and } \frac{\partial w}{\partial x}(5,t) = \beta \text{ for } t \geq 0.$$

Solution. The problem for an equilibrium solution is

$$0 = u''(x) + 1$$

$$u'(0) = 1 \text{ and } u'(5) = \beta.$$

From the DE we get

$$\int_0^5 u''(x)dx = \int_0^5 (-1)dx$$

so

$$u'(5) - u'(0) = -5 \text{ or } \beta - 1 = -5.$$

Thus

$$\beta = -4.$$

3. Find the equilibrium solution with no undetermined constants in the previous problem..

Solution.. The problem for an equilibrium solution is

$$0 = u''(x) + 1,$$

$$u'(0) = 1 \text{ and } u'(5) = -4.$$

From the DE we get

$$u'(x) = -x + c_1$$

then

$$u(x) = -\frac{1}{2}x^2 + c_1 x + c_2.$$

From $u'(0) = 1$ it follows that

$$c_1 = 1$$

so

$$u(x) = -\frac{1}{2}x^2 + x + c_2.$$

To find c_2 we first show that $\int_0^5 w(x,t)dx$ is constant in t.

$$\begin{aligned}
\frac{d}{dt}\int_0^5 w(x,t)dx &= \int_0^5 \frac{\partial w}{\partial t}(x,t)dx \\
&= \int_0^5 \frac{\partial^2 w}{\partial x^2}(x,t) + 1 \ dx \\
&= \frac{\partial w}{\partial x}(5,t) - \frac{\partial w}{\partial x}(0,t) + 5 \\
&= -4 - 1 + 5 = 0.
\end{aligned}$$

So $\int_0^5 w(x,t)dx$ is constant in t.Thus

$$\begin{aligned}\int_0^5 w(x,0)dx &= \int_0^5 w(x,t)dx(\text{any } t) = \lim_{t\to\infty}\int_0^5 w(x,t)dx \\ &= \int_0^5 \lim_{t\to\infty} w(x,t)dx = \int_0^5 u(x)dx.\end{aligned}$$

From this it follows that

$$\int_0^5 xdx = \int_0^5 (-\frac{1}{2}x^2 + x + c_2)dx$$

or

$$\frac{25}{2} = -\frac{25}{3} + 5c_2.$$

So

$$c_2 = \frac{25}{6}$$

and

$$u(x) = -\frac{1}{2}x^2 + x + \frac{25}{6}.$$

4. Consider the following two-point boundary value problem in which L is a positive number.

$$\begin{array}{ll} \text{(i)} & -\varphi''(x) = \lambda\varphi(x) \qquad \text{for } 0 \le x \le L, \\ \text{(ii)} & \varphi(0) = 0, \text{ and} \\ \text{(iii)} & \varphi(L) + \varphi'(L) = 0. \end{array}$$

Use the Rayleigh Quotient to show that all eigenvalues are non negative.

Solution. Suppose that λ is an eigenvalue and φ is a corresponding eigenfunction. Then

$$\begin{aligned}\lambda &= \frac{\varphi(0)\varphi'(0) - \varphi(L)\varphi'(L) + \int_0^L(\varphi'(x))^2dx}{\int_0^L(\varphi(x))^2dx} \\ &= \frac{0\cdot\varphi'(0) - \varphi(L)(-\varphi(L)) + \int_0^L(\varphi'(x))^2dx}{\int_0^L(\varphi(x))^2dx} \\ &= \frac{(\varphi(L))^2 + \int_0^L(\varphi'(x))^2dx}{\int_0^L(\varphi(x))^2dx} \ge 0\end{aligned}$$

5. Is the number zero an eigenvalue for the two-point boundary value problem in Problem 4? Explain why or why not.

Solution Suppose that φ is a solution to (i), (ii), and (iii) when $\lambda = 0$. From (i),

$$\varphi'(x) = c_1$$

and

$$\varphi(x) = c_1 x + c_2.$$

Then from (ii),

$$c_2 = 0,$$

and from (iii),

$$c_1 L + c_1 = 0 \text{ implying } (L+1)c_1 = 0$$

so

$$c_1 = 0.$$

Thus

$$\varphi(x) = 0 \text{ for } 0 \le x \le L.$$

Since the only solution is the zero function, the number zero is not an eigenvalue.

6. For the two-point boundary value problem given in Problem 4, find the matrix $D(\lambda)$ and the determinant $\Delta(\lambda)$ in the case where $\lambda > 0$.

Solution. The boundary conditions are equivalent to

$$\beta_1 \varphi(0) + \beta_2 \varphi'(0) = 0$$

and

$$\beta_3 \varphi(L) + \beta_4 \varphi'(L) = 0$$

where $\beta_1 = 1$, $\beta_2 = 0$, $\beta_3 = 1$, and $\beta_4 = 1$ so

$$D(\lambda) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \Phi_\lambda(0) + \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} \Phi_\lambda(L)$$

where

$$\Phi_\lambda(x) = \begin{pmatrix} \cos\sqrt{\lambda}x & \sin\sqrt{\lambda}x \\ -\sqrt{\lambda}\sin\sqrt{\lambda}x & \sqrt{\lambda}\cos\sqrt{\lambda}x \end{pmatrix}.$$

$$D(\lambda) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{\lambda} \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \cos\sqrt{\lambda}L & \sin\sqrt{\lambda}L \\ -\sqrt{\lambda}\sin\sqrt{\lambda}L & \sqrt{\lambda}\cos\sqrt{\lambda}L \end{pmatrix},$$

so

$$D(\lambda) = \begin{pmatrix} 1 & 0 \\ \cos L\sqrt{\lambda} - \sqrt{\lambda}\sin L\sqrt{\lambda} & \sin L\sqrt{\lambda} + \sqrt{\lambda}\cos L\sqrt{\lambda} \end{pmatrix}$$

and

$$\Delta(\lambda) = \det D(\lambda) = \sin L\sqrt{\lambda} + \sqrt{\lambda}\cos L\sqrt{\lambda}.$$

7. Suppose that $\mathcal{D}$ is a region in the plane with the property that if each of (x_1, y_1) and (x_2, y_2) is in $\mathcal{D}$ then each of (x_1, y_2) and (x_2, y_1) is in $\mathcal{D}$. Suppose that $F(x) = G(y)$ whenever (x, y) is in $\mathcal{D}$. Show that there is a constant C such that if (x, y) is in $\mathcal{D}$, then

$$F(x) = C = G(y).$$

Solution. Let (x_0, y_0) be a point in $\mathcal{D}$ and let

$$C = F(x_0) = G(y_0).$$

Suppose that (x, y) is a point in $\mathcal{D}$. Then each of (x, y) and (x_0, y_0) is in $\mathcal{D}$. Since $(x, y_0)$is in $\mathcal{D}$, it follows that

$$F(x) = G(y_0) = C.$$

Since (x_0, y) is in $\mathcal{D}$, it follows that

$$F(x_0) = G(y) = C.$$

Thus

$$F(x) = C = G(y).$$

8. Suppose that $\{\phi_k\}_{k=1}^n$ is orthogonal on $[0, L]$ and $< \phi_k, \phi_k > \neq 0$ for $k = 1, \dots, n$. Suppose that $f = \sum_{k=1}^n c_k \phi_k$. Derive a formula that gives c_k in terms of f, ϕ_k, and the inner product. Suggestion: Note that the summation index can be changed. For example,

$$f = \sum_{j=1}^n c_j \phi_j$$

Solution. For $k = 1, \dots, n$,

$$< f, \varphi_k > = < \sum_{j=1}^n c_j \phi_j, \varphi_k > = \sum_{j=1}^n c_j < \phi_j, \varphi_k > .$$

Since $< \phi_j, \varphi_k > = 0$ when $j \neq k$,

$$< f, \varphi_k > = c_k < \phi_k, \varphi_k >$$

so

$$c_k = \frac{< f, \varphi_k >}{< \phi_k, \varphi_k >}.$$

9. Suppose that each of L and κ is a positive number.

(a) Suppose that

$$\frac{\partial u}{\partial t}(x,t) = \kappa \frac{\partial^2 u}{\partial x^2}(x,t) \text{ for } t \geq 0 \text{ and } 0 \leq x \leq L,$$

that

$$u(x,t) = \varphi(x)G(t),$$

and that

$$u(x,t) \neq 0$$

for $0 \leq x \leq L$ and $t \geq 0$. Derive ordinary differential equations for φ and G.

Solution. From the PDE, it follows that

$$\varphi(x)G'(t) = \kappa\varphi''(x)G(t).$$

Dividing each side of this equation by $\kappa\varphi(x)G(t)$ produces

$$\frac{G'(t)}{\kappa G(t)} = \frac{\varphi''(x)}{\varphi(x)}.$$

Since this is true for $t \geq 0$ and $0 \leq x \leq L$, it follows that there is a constant C such that

$$\frac{G'(t)}{\kappa G(t)} = C = \frac{\varphi''(x)}{\varphi(x)}.$$

Renaming C to be $-\lambda$ it follows that

$$-\varphi''(x) = \lambda\varphi(x) \text{ for } 0 \leq x \leq L$$

and

$$G'(t) + \kappa\lambda G(t) = 0 \text{ for } t \geq 0.$$

(b) Suppose that

$$u(x,t) = \varphi(x)G(t) \text{ for } t \geq 0 \text{ and } 0 \leq x \leq L,$$

$$\beta_1 u(0,t) + \beta_2 \frac{\partial u}{\partial x}(0,t) = 0,$$

and

$$\beta_3 u(L,t) + \beta_4 \frac{\partial u}{\partial x}(L,t) = 0$$

for $t \geq 0$. Also suppose that

$$u(x_0,t_0) \neq 0$$

for some (x_0,t_0). Show that

$$\beta_1\varphi(0) + \beta_2\varphi'(0) = 0$$

and

$$\beta_3\varphi(L) + \beta_4\varphi'(L) = 0.$$

Solution. We have

$$\beta_1\varphi(0)G(t) + \beta_2\varphi'(0)G(t) = 0,$$

and

$$\beta_3\varphi(L)G(t) + \beta_4\varphi'(L)G(t) = 0$$

for $t \geq 0$ so

$$\beta_1\varphi(0)G(t_0) + \beta_2\varphi'(0)G(t_0) = 0, \tag{1}$$

and

$$\beta_3\varphi(L)G(t_0) + \beta_4\varphi'(L)G(t_0) = 0. \tag{2}$$

Since $u(x_0, t_0) \neq 0$, it follows that $G(t_0) \neq 0$, and dividing each side of (1) and (2) by $G(t_0)$ produces

$$\beta_1\varphi(0) + \beta_2\varphi'(0) = 0$$

and

$$\beta_3\varphi(L) + \beta_4\varphi'(L) = 0.$$

10. Establish the following convergence results.

    (a) Evaluate

$$\sum_{k=1}^{\infty} e^{-kt}.$$

when t is a positive number. Then show that

$$\sum_{k=1}^{\infty} e^{-k^2 t}$$

exists and is finite when t is a positive number.

Solution.

$$\sum_{k=0}^{\infty} r^k = \frac{1}{1-r} \text{ when } -1 < r < 1$$

so

$$\sum_{k=1}^{\infty} r^k = \frac{1}{1-r} - 1 = \frac{r}{1-r} \text{ when } -1 < r < 1.$$

$$0 < e^{-t} < 1 \text{ when } t > 0$$

so

$$\sum_{k=1}^{\infty} e^{-kt} = \sum_{k=1}^{\infty} (e^{-t})^k = \frac{e^{-t}}{1 - e^{-t}}$$

Since

$$0 \le e^{-k^2 t} \le e^{-kt}$$

when $t > 0$, it follows from the comparison test that

$$\sum_{k=1}^{\infty} e^{-k^2 t}$$

exists and is finite.

(b) Suppose that $\{c_k\}$ is a bounded sequence of numbers and

$$S_n(x,t) = \sum_{k=1}^{n} c_k (\sin kx) e^{-k^2 t}$$

for $0 \le x \le \pi$, $t > 0$ and $n = 1, 2, \ldots$. Suppose that t_0 is a positive number and $\mathcal{D}$ is the set consisting of all (x,t) where $0 \le x \le L$ and $t \ge t_0$. Show that $\{S_n\}$ converges uniformly on $\mathcal{D}$.

Solution. Let B be a bound for the sequence $\{c_k\}$. Then

$$|c_k (\sin kx) e^{-k^2 t}| \le |c_k| \cdot |\sin kx| \cdot |e^{-k^2 t}| \le B \cdot 1 \cdot e^{-k^2 t} \le B e^{-k^2 t_0}$$

for all (x,t) in $\mathcal{D}$.

$$\sum_{k=1}^{\infty} e^{-k^2 t_0}$$

hence

$$\sum_{k=1}^{\infty} B e^{-k^2 t_0}$$

exists and is finite by Part a, so the uniform convergence of $\{S_n\}$ follows from the Weierstrass M-Test with

$$M_k = B e^{-k^2 t_0}.$$