Math 3363 Examination II Solutions

Spring 2020

Upload your solutions in a pdf file by 11:59 p.m. Thursday, April 2. You may use your text, notes, and the material posted on Dr. Walker's web site, but you must do your own work.

You may use the following information without derivation.

- A proper listing of eigenvalues and eigenfunctions for

$$\begin{array}{ll} \text{(i)} & -\varphi''(x) = \lambda\varphi(x) \quad \text{for } 0 \le x \le L, \\ \text{(ii)} & \varphi(0) = 0, \text{ and} \\ \text{(iii)} & \varphi(L) = 0 \end{array}$$

  is $\{\lambda_k\}_{k=1}^{\infty}$ and $\{\varphi_k\}_{k=1}^{\infty}$ where $\lambda_k = (\frac{k\pi}{L})^2$ and $\varphi_k(x) = \sin\frac{k\pi x}{L}$.

- A proper listing of eigenvalues and eigenfunctions for

$$\begin{array}{ll} \text{(i)} & -\varphi''(x) = \lambda\varphi(x) \quad \text{for } 0 \le x \le L, \\ \text{(ii)} & \varphi'(0) = 0, \text{ and} \\ \text{(iii)} & \varphi'(L) = 0 \end{array}$$

  is $\{\lambda_k\}_{k=0}^{\infty}$ and $\{\varphi_k\}_{k=0}^{\infty}$ where $\lambda_k = (\frac{k\pi}{L})^2$ and $\varphi_k(x) = \cos\frac{k\pi x}{L}$. Note that $\lambda_0 = 0$ and $\varphi_0(x) = 1$.

- A proper listing of eigenvalues and eigenfunctions for

$$\begin{array}{ll} \text{(i)} & -\varphi''(x) = \lambda\varphi(x) \quad \text{for } 0 \le x \le L, \\ \text{(ii)} & \varphi(0) = 0, \text{ and} \\ \text{(iii)} & \varphi'(L) = 0 \end{array}$$

  is $\{\lambda_k\}_{k=1}^{\infty}$ and $\{\varphi_k\}_{k=1}^{\infty}$ where $\lambda_k = \left(\frac{(2k-1)\pi}{2L}\right)^2$ and $\varphi_k(x) = \sin\frac{(2k-1)\pi x}{2L}$.

- A proper listing of eigenvalues and eigenfunctions for

$$\begin{array}{ll} \text{(i)} & -\varphi''(x) = \lambda\varphi(x) \quad \text{for } 0 \le x \le L, \\ \text{(ii)} & \varphi'(0) = 0, \text{ and} \\ \text{(iii)} & \varphi(L) = 0 \end{array}$$

  is $\{\lambda_k\}_{k=1}^{\infty}$ and $\{\varphi_k\}_{k=1}^{\infty}$ where $\lambda_k = \left(\frac{(2k-1)\pi}{2L}\right)^2$ and $\varphi_k(x) = \cos\frac{(2k-1)\pi x}{2L}$.

- A proper listing of eigenvalues and eigenfunctions for

$$\begin{array}{lll} \text{(i)} & -\varphi''(x) = \lambda\varphi(x) & \text{for } -L \le x \le L, \\ \text{(ii)} & \varphi(-L) = \varphi(L), \text{ and} & \\ \text{(iii)} & \varphi'(-L) = \varphi'(L) & \end{array}$$

is $\{\lambda_k\}_{k=0}^{\infty}$ and $\{\varphi_k\}_{k=0}^{\infty}$ where

$$\lambda_0 = 0,\ \varphi_0(x) = 1 \text{ for } -L \le x \le L,$$

$$\begin{aligned} \lambda_{2k-1} &= \lambda_{2k} = (\frac{k\pi}{L})^2 \text{ for } k = 1, 2, 3, \dots, \\ \varphi_{2k-1}(x) &= \cos\frac{k\pi}{L}x, \text{ and } \varphi_{2k}(x) = \sin\frac{k\pi}{L}x \text{ for } k = 1, 2, 3, \dots \text{ and } -L \le x \le L. \end{aligned}$$

For this problem, the eigenvalues are zero and $(\frac{k\pi}{L})^2$ for $k = 1, 2, 3, \dots$. The eigenspace corresponding to zero is one dimensional and a corresponding eigenfunction is the constant function with value one. For $k = 1, 2, 3, \dots$, the eigenspace corresponding to $(\frac{k\pi}{L})^2$ is two dimensional and a corresponding orthogonal pair of eigenfunctions is the pair of functions whose values at x are $\cos\frac{k\pi}{L}x$ and $\sin\frac{k\pi}{L}x$.

1. Find the function v of the form

$$v(x, y) = ax + by + cxy + d$$

such that

$$\begin{aligned} v(0,0) &= 3, \\ v(4,0) &= -5, \\ v(4,2) &= 2, \text{ and} \\ v(0,2) &= -1. \end{aligned}$$

Solution.

$$v(0,0) = a\cdot 0 + b\cdot 0 + c\cdot 0\cdot 0 + d = 3 \text{ so } d = 3$$

$$v(4,0) = a\cdot 4 + b\cdot 0 + c\cdot 4\cdot 0 + 3 = -5 \text{ so } a = -2$$

$$v(0,2) = a\cdot 0 + b\cdot 2 + c\cdot 0\cdot 2 + 3 = -1 \text{ so } b = -2$$

$$v(4,2) = -2\cdot 4 - 2\cdot 2 + c\cdot(4)(2) + 3 = 2 \text{ so } c = \frac{11}{8}$$

$$v(x,y) = -2x - 2y + \frac{11}{8}xy + 3$$

2. A hollow cylinder of height H and circumference $2L$ is open at both ends and is in thermal equilibrium. It is insulated except around its top and bottom rim. The temperature at each point on the top and bottom rims is known. Find the temperature at each point of the cylinder.

Think of a plate lying in the rectangle $[-L, L] \times [0, H]$ being formed in to the cylinder by joining the left and right edges. This is the problem to solve:

$$\begin{aligned}
\frac{\partial^2 u}{\partial x^2}(x.y) + \frac{\partial^2 u}{\partial y^2}(x.y) &= 0 \text{ for } -L \le x \le L \text{ and } 0 \le y \le H && (1)\\
u(-L, y) &= u(L, y) \text{ for } 0 \le y \le H && (2)\\
\frac{\partial u}{\partial x}(-L, y) &= \frac{\partial u}{\partial x}(L, y) \text{ for } 0 \le y \le H && (3)\\
u(x, 0) &= f(x) \text{ for } -L \le x \le L && (4)\\
u(x, H) &= g(x) \text{ for } -L \le x \le L && (5)
\end{aligned}$$

Solution. The solution u to (1)-(5) is given by

$$u = u^{\text{lower}} + u^{\text{upper}}$$

where u^{lower} is the solution to

$$\begin{aligned}
\frac{\partial^2 u}{\partial x^2}(x.y) + \frac{\partial^2 u}{\partial y^2}(x.y) &= 0 \text{ for } -L \le x \le L \text{ and } 0 \le y \le H && (6)\\
u(-L, y) &= u(L, y) \text{ for } 0 \le y \le H && (7)\\
\frac{\partial u}{\partial x}(-L, y) &= \frac{\partial u}{\partial x}(L, y) \text{ for } 0 \le y \le H && (8)\\
u(x, 0) &= f(x) \text{ for } -L \le x \le L && (9)\\
u(x, H) &= 0 \text{ for } -L \le x \le L && (10)
\end{aligned}$$

and u^{upper} is the solution to

$$\begin{aligned}
\frac{\partial^2 u}{\partial x^2}(x.y) + \frac{\partial^2 u}{\partial y^2}(x.y) &= 0 \text{ for } -L \le x \le L \text{ and } 0 \le y \le H && (11)\\
u(-L, y) &= u(L, y) \text{ for } 0 \le y \le H && (12)\\
\frac{\partial u}{\partial x}(-L, y) &= \frac{\partial u}{\partial x}(L, y) \text{ for } 0 \le y \le H && (13)\\
u(x, 0) &= 0 \text{ for } -L \le x \le L && (14)\\
u(x, H) &= g(x) \text{ for } -L \le x \le L && (15)
\end{aligned}$$

Solution for u^{lower}. Suppose that u is an elementary separated solution to (6). This means

$$u(x, y) = \varphi(x)h(y)$$

for some pair of one-place functions φ and h. Inserting this into (6), we have

$$\varphi''(x)h(y) + \varphi(x)h''(y) = 0. \qquad (16)$$

Assuming for now that

$$u(x.y) \neq 0,$$

and dividing each side of (16) by $\varphi(x)h(y)$, we have

$$\frac{\varphi''(x)h(y)}{\varphi(x)h(y)} + \frac{\varphi(x)h''(y)}{\varphi(x)h(y)} = 0,$$

so

$$\frac{h''(y)}{h(y)} = -\frac{\varphi''(x)}{\varphi(x)}.$$

This holds for all y with $0 \leq y \leq H$ and x with $-L \leq x \leq L$, so there is a constant λ such that

$$\frac{h''(y)}{h(y)} = \lambda = -\frac{\varphi''(x)}{\varphi(x)} \tag{17}$$

for all y with $0 \leq y \leq H$ and x with $-L \leq x \leq L$. From (17) we then have

$$-\varphi''(x) = \lambda\varphi(x) \text{ for all } x \text{ in } [-L, L] \tag{18}$$

and

$$h''(y) = \lambda h(y) \text{ for all } y \text{ in } [0, H]. \tag{19}$$

It is worth noting that if

$$u(x, y) = \varphi(x)h(y)$$

and (18) and (19) hold, then

$$\begin{aligned} \frac{\partial^2 u}{\partial x^2}(x, y) &= \varphi''(x)h(y) = -\lambda\varphi(x)h(y) \\ &= -\varphi(x)h''(y) = -\frac{\partial^2 u}{\partial y^2}(x, y) \end{aligned}$$

so the PDE (6)

$$\frac{\partial^2 u}{\partial x^2}(x, y) + \frac{\partial^2 u}{\partial y^2}(x.y) = 0$$

will be satisfied, and we no longer need to assume that $u(x, y) \neq 0$. Continuing with our assumption that

$$u(x, y) = \varphi(x)h(y)$$

we have from conditions (7) and (8) that either $h(y) = 0$ for all y in $[0, H]$ which we reject because of (9) or

$$\varphi(-L) = \varphi(L) \tag{20}$$

and

$$\varphi'(-L) = \varphi'(L) \tag{21}$$

which we must accept. In a similar way we have from (10) that

$$h(H) = 0 \tag{22}$$

The problem consisting of (18), (20), and (21) is one which we have studied. The eigenvalues are zero and $(\frac{k\pi}{L})^2$ for $k = 1, 2, 3, \ldots$. The eigenspace corresponding to zero is one dimensional and a corresponding eigenfunction is the constant function with value one. For $k = 1, 2, 3, \ldots$, the eigenspace corresponding to $(\frac{k\pi}{L})^2$ is two dimensional and a corresponding orthogonal pair of eigenfunctions is the pair of functions whose values at x are $\cos\frac{k\pi}{L}x$ and $\sin\frac{k\pi}{L}x$.

When $\lambda = 0$, the solutions to (19) are linear combinations of the functions whose values at y are $H - y$ and ,and the solutions to (19) and (22) are multiples of the function whose value at y is $H - y$. Corresponding to $\lambda = 0$ we have the elementary separated solution whose value at (x, y) is

$$1 \cdot (H - y) = H - y.$$

When $\lambda = (\dfrac{k\pi}{L})^2$, the solutions fo (19) are linear combinations of sthe functions whose values at y are $\sinh\dfrac{k\pi y}{L}$ and $\sinh\dfrac{k\pi}{L}(H - y)$, and the solutions to (19) and (22) are multiples of the function whose value at y is $\sinh\dfrac{k\pi}{L}(H - y)$. Corresponding to $\lambda = (\dfrac{k\pi}{L})^2$, we have elementary sepsrated solutions whose values at (x, y) are

$$\cos\frac{k\pi x}{L}\sinh\frac{k\pi}{L}(H - y) \text{ and } \sin\frac{k\pi x}{L}\sinh\frac{k\pi}{L}(H - y)$$

The problem consisting of (6), (7), (8), and (10) is linear and homogeneous so anything of the form

$$A_0(H - y) + \sum_{k=1}^{n}[A_k\cos\frac{k\pi x}{L} + B_k\sin\frac{k\pi x}{L}]\sinh\frac{k\pi}{L}(H - y)$$

will be a solution. In order for (9) to be satisfied, an infinite sum will be needed, so we conjecture that the solution u to (6)-(10) is of the form

$$u(x, y) = A_0(H - y) + \sum_{k=1}^{\infty}[A_k\cos\frac{k\pi x}{L} + B_k\sin\frac{k\pi x}{L}]\sinh\frac{k\pi}{L}(H - y).$$

Condition (9) will hold if and only if

$$\begin{aligned} f(x) &= A_0 H + \sum_{k=1}^{\infty}[A_k\cos\frac{k\pi x}{L} + B_k\sin\frac{k\pi x}{L}]\sinh\frac{k\pi}{L}H \\ &= A_0 H + \sum_{k=1}^{\infty}[(A_k\sinh\frac{k\pi}{L}H)\cos\frac{k\pi x}{L} + (B_k\sinh\frac{k\pi}{L}H)\sin\frac{k\pi x}{L}]. \end{aligned}$$

The function f is being expressed as the limit of a Fourier series. Thus

$$A_0 H = \frac{1}{2L}\int_{-L}^{L} f(x)dx \text{ so } A_0 = \frac{1}{2HL}\int_{-L}^{L} f(x)dx,$$

$$A_k \sinh \frac{k\pi}{L} H = \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{k\pi x}{L} dx \text{ so } A_k = \frac{1}{L \sinh \dfrac{k\pi H}{L}} \int_{-L}^{L} f(x) \cos \frac{k\pi x}{L} dx,$$

and

$$B_k \sinh \frac{k\pi}{L} H = \frac{1}{L} \int_{-L}^{L} f(x) \sin \frac{k\pi x}{L} dx \text{ so } B_k = \frac{1}{L \sinh \dfrac{k\pi H}{L}} \int_{-L}^{L} f(x) \sin \frac{k\pi x}{L} dx$$

for $k = 1, 2, \ldots$.

Solution for u^{upper}. The derivation is similar to that for u^{lower}. $H - y$ gets replaced by y.The solution u is given by

$$u(x, y) = A_0 y + \sum_{k=1}^{\infty} [A_k \cos \frac{k\pi x}{L} + B_k \sin \frac{k\pi x}{L}] \sinh \frac{k\pi}{L} y.$$

where

$$A_0 H = \frac{1}{2L} \int_{-L}^{L} g(x) dx \text{ so } A_0 = \frac{1}{2HL} \int_{-L}^{L} g(x) dx,$$

$$A_k \sinh \frac{k\pi}{L} H = \frac{1}{L} \int_{-L}^{L} g(x) \cos \frac{k\pi x}{L} dx \text{ so } A_k = \frac{1}{L \sinh \dfrac{k\pi H}{L}} \int_{-L}^{L} g(x) \cos \frac{k\pi x}{L} dx,$$

and

$$B_k \sinh \frac{k\pi}{L} H = \frac{1}{L} \int_{-L}^{L} g(x) \sin \frac{k\pi x}{L} dx \text{ so } B_k = \frac{1}{L \sinh \dfrac{k\pi H}{L}} \int_{-L}^{L} g(x) \sin \frac{k\pi x}{L} dx.$$

$$A_0 = \frac{1}{2HL} \int_{-L}^{L} g(x) dx,$$

$$A_k = \frac{1}{L \sinh \dfrac{k\pi H}{L}} \int_{-L}^{L} g(x) \cos \frac{k\pi x}{L} dx,$$

and

$$B_k = \frac{1}{L \sinh \dfrac{k\pi H}{L}} \int_{-L}^{L} g(x) \sin \frac{k\pi x}{L} dx.$$

Solution to the Original Problem. The solution to (1)-(5) is u where

$$\begin{aligned} u(x, y) &= A_0 (H - y) + \sum_{k=1}^{\infty} [A_k \cos \frac{k\pi x}{L} + B_k \sin \frac{k\pi x}{L}] \sinh \frac{k\pi}{L} (H - y) \\ &\quad + C_0 y + \sum_{k=1}^{\infty} [C_k \cos \frac{k\pi x}{L} + D_k \sin \frac{k\pi x}{L}] \sinh \frac{k\pi}{L} \end{aligned}$$

in which

$$A_0 = \frac{1}{2HL}\int_{-L}^{L} f(x)dx,$$

$$A_k = \frac{1}{L\sinh\dfrac{k\pi H}{L}}\int_{-L}^{L} f(x)\cos\frac{k\pi x}{L}dx,$$

$$B_k = \frac{1}{L\sinh\dfrac{k\pi H}{L}}\int_{-L}^{L} f(x)\sin\frac{k\pi x}{L}dx,$$

$$C_0 = \frac{1}{2HL}\int_{-L}^{L} g(x)dx,$$

$$C_k = \frac{1}{L\sinh\dfrac{k\pi H}{L}}\int_{-L}^{L} g(x)\cos\frac{k\pi x}{L}dx,$$

and

$$D_k = \frac{1}{L\sinh\dfrac{k\pi H}{L}}\int_{-L}^{L} g(x)\sin\frac{k\pi x}{L}dx.$$

3. Derive the solution to the following problem for Laplace's equation in a rectangle.

$$\begin{aligned}
\frac{\partial^2 u}{\partial x^2}(x.y) + \frac{\partial^2 u}{\partial y^2}(x.y) &= 0 \text{ for } 0 \le x \le L \text{ and } 0 \le y \le H &(1)\\
\frac{\partial u}{\partial x}(0,y) &= 0 \text{ for } 0 \le y \le H &(2)\\
u(L,y) &= 0 \text{ for } 0 \le y \le H &(3)\\
u(x,H) &= 0 \text{ for } 0 \le x \le L &(4)\\
u(x,0) &= f(x) \text{ for } 0 \le x \le L &(5)
\end{aligned}$$

Solution. Suppose that u is an elementary separated solution to (1). This means

$$u(x,y) = \varphi(x)h(y)$$

for some pair of one-place functions φ and h. Inserting this into (1), we have

$$\varphi''(x)h(y) + \varphi(x)h''(y) = 0.$$

Assuming for now that

$$u(x.y) \neq 0,$$

and dividing each side by $\varphi(x)h(y)$, we have

$$\frac{\varphi''(x)h(y)}{\varphi(x)h(y)} + \frac{\varphi(x)h''(y)}{\varphi(x)h(y)} = 0,$$

so

$$\frac{h''(y)}{h(y)} = -\frac{\varphi''(x)}{\varphi(x)}.$$

This holds for all y with $0 \leq y \leq H$ and x with $0 \leq x \leq L$, so there is a constant λ such that

$$\frac{h''(y)}{h(y)} = \lambda = -\frac{\varphi''(x)}{\varphi(x)}$$

for all y with $0 \leq y \leq H$ and x with $0 \leq x \leq L$. We then have

$$-\varphi''(x) = \lambda\varphi(x) \text{ for all } x \text{ in } [0, L] \tag{6}$$

and

$$h''(y) = \lambda h(y) \text{ for all } y \text{ in } [0, H]. \tag{7}$$

It is worth noting that if

$$u(x, y) = \varphi(x)h(y)$$

and (6) and (7) hold, then

$$\begin{aligned} \frac{\partial^2 u}{\partial x^2}(x, y) &= \varphi''(x)h(y) = -\lambda\varphi(x)h(y) \\ &= -\varphi(x)h''(y) = -\frac{\partial^2 u}{\partial y^2}(x, y) \end{aligned}$$

so the PDE (1)

$$\frac{\partial^2 u}{\partial x^2}(x, y) + \frac{\partial^2 u}{\partial y^2}(x.y) = 0$$

will be satisfied, and we no longer need to assume that $u(x, y) \neq 0$. Continuing with our assumption that

$$u(x, y) = \varphi(x)h(y)$$

we have from conditions (2) and (3) that either $h(y) = 0$ for all y in $[0, H]$ which we reject because of (5) or

$$\varphi'(0) = 0 \tag{8}$$

and

$$\varphi(L) = 0 \tag{9}$$

which we must accept. In a similar way we have from (4) (which stated that $u(x, H) = 0$) that

$$h(H) = 0 \tag{10}$$

The Sturm-Louville problem consisting of (6), (8), and (9)is one which we have studied. A proper listing of eigenvalues and eigenfunctions for this problem is

$$\{\lambda_k\}_{k=1}^{\infty} \text{ and } \{\varphi_k\}_{k=1}^{\infty}$$

where

$$\lambda_k = (\frac{(2k-1)\pi}{2L})^2 \text{ for } k = 1, 2, \ldots$$

and

$$\varphi_k(x) = \sin \rho_k x \text{ for all } x \text{ in } [0, L] \text{ and } k = 1, 2, \ldots \text{. where } \rho_k = \frac{(2k-1)\pi}{2L}$$

The equation (7)

$$h''(y) = \lambda h(y)$$

is equivalent to

$$h''(y) - \lambda h(y) = 0. \tag{11}$$

When $\lambda > 0$ as it must be because all eigenvalues for the problem (6), (8), and (8) are positive, a linearly independent pair of solutions to (14) is the pair whose values at y are

$$\sinh \sqrt{\lambda} y \text{ and } \sinh \sqrt{\lambda}(H - y).$$

We have

$$h(y) = c_1 \sinh \sqrt{\lambda} y + c_2 \sinh \sqrt{\lambda}(H - y).$$

We have from (10) that $h(H) = 0$, so

$$c_1 \sinh \lambda H + c_2 \sinh \sqrt{\lambda}(H - H) = 0,$$

Using the fact that $\sinh 0 = 0$ and $\sinh z \neq 0$ when $z \neq 0$, we have that $c_1 = 0$ and see that when $\lambda = \lambda_k$ then the solutions to (10) and (11) are constant multiples of h_k where

$$h_k(y) = \sinh \rho_k (H - y).$$

The problem consisting of (1), (2), (3), and (4) is linear and homogeneous, so if $\{E_k\}_{k=1}^{n}$is a finite sequence of numbers and

$$u(x, y) = \sum_{k=1}^{n} E_k \varphi_k(x) h_k(y),$$

then u will be a solution to (1), (2), (3), and (4). Thus we hope that the solution to the problem consisting of (1) through (5) will be of the form

$$u(x, y) = \sum_{k=1}^{\infty} E_k \varphi_k(x) h_k(y)$$

for some perhaps infinite sequence of constants $\{E_k\}_{k=1}^{\infty}$.Condition (5)

$$u(x, 0) = f(x) \text{ for } x \text{ in } [0, L],$$

implies

$$f = \sum_{k=1}^{\infty} E_k \varphi_k h_k(0) = \sum_{k=1}^{\infty} (E_k \sinh \rho_k H) \varphi_k.$$

Since $\{\varphi_k\}_{k=1}^{\infty}$ is an orthogonal sequence of non zero function this implies

$$(E_k \sinh \rho_k H) = \frac{< f, \varphi_k >}{< \varphi_k, \varphi_k >}$$

so

$$E_k = \frac{< f, \varphi_k >}{\sinh \rho_k H < \varphi_k, \varphi_k >}$$

for $k = 1, 2, \ldots$ where the inner product is defined by

$$< \alpha, \beta >= \int_0^L \alpha(x)\beta(x)dx.$$

For this sequence $\{\varphi_k\}$,

$$< \varphi_k, \varphi_k >= \int_0^L (\cos \rho_k)^2 dx = \frac{L}{2} \text{ for } k = 1, 2, \ldots.$$

In summary, the solution to the original problem (1) through (5) is u where

$$u(x, y) = \sum_{k=1}^{\infty} E_k \cos \rho_k x \sinh \rho_k (H - y)$$

in which

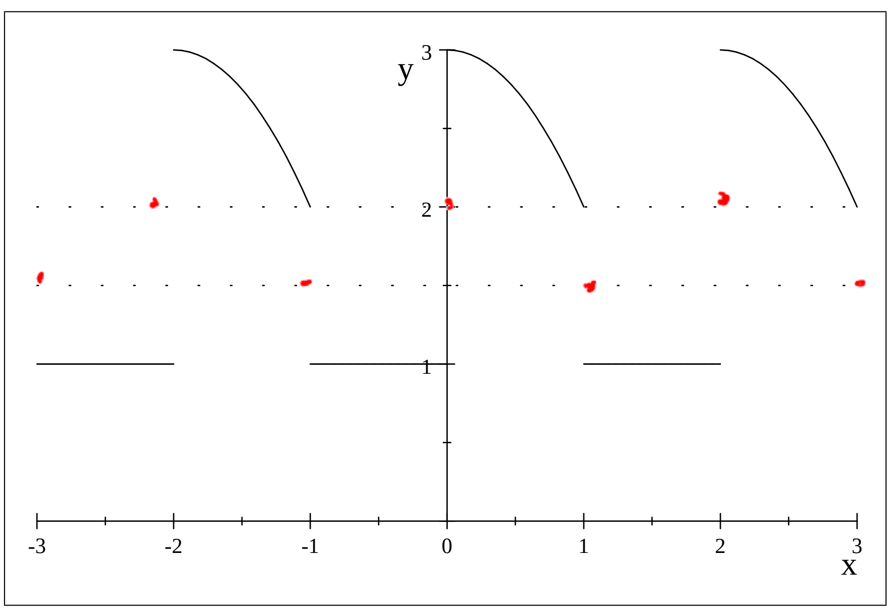
$$E_k = \frac{2}{L \sinh \rho_k H} \int_0^L f(x) \cos \rho_k dx \text{ for } k = 1, 2, \ldots$$

4. Let f be given by

$$f(x) = \begin{cases} 1 & if \quad -1 \leq x < 0 \\ 3 - x^2 & if \quad 0 \leq x \leq 1 \end{cases}.$$

Let h be the limit of the Fourier Series ($L = 1$) for f. Sketch the graph of h over $[-3, 3]$. Be sure to show the value of h at each number in $[-3, 3]$.

Solution.



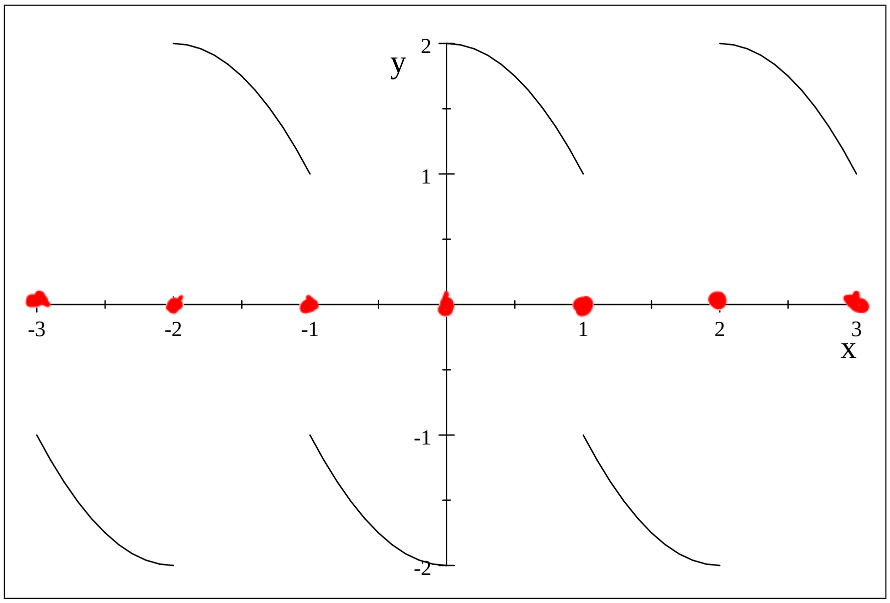
h

5. Let f be given by

$$f(x) = 2 - x^2 \text{ for } 0 \le x \le 1.$$

(a) Let g be the limit of the sine series ($L = 1$) for f. Sketch the graph of g over $[-3, 3]$. Be sure to show the value of g at each number in $[-3, 3]$.

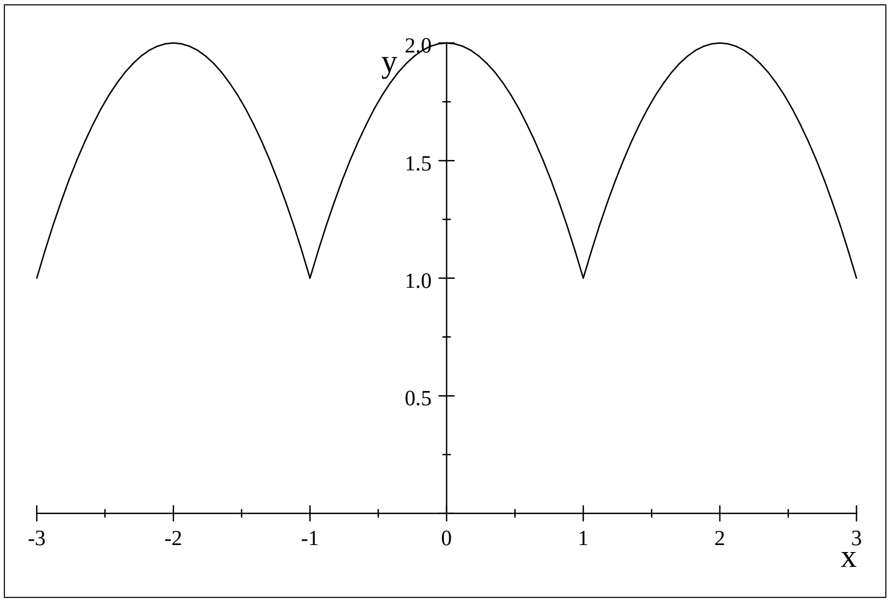
Solution.



g

(b) Let h be the limit of the cosine series ($L = 1$) for f. Sketch the graph of h over $[-3, 3]$. Be sure to show the value of h at each number in $[-3, 3]$.

Solution.



h

6. Let

$$f(x) = \begin{cases} -1 & if \quad -1 \le x < 0 \\ 1 & if \quad 0 \le x \le 1 \end{cases}$$

and let $\{S_n\}_{n=1}^{\infty}$ be the Fourier Series for f. Does $\{S_n\}$ converge uniformly? Explain why or why not.

Solution. Each S_n is continuous, but the limit function is not, so the convergence cannot be uniform.

7. Derive the solution to the following wave equation problem.

$$\frac{\partial^2 u}{\partial t^2}(x,t) = c^2 \frac{\partial^2 u}{\partial x^2}(x,t) \text{ for } 0 \le x \le L \text{ and all } t \text{ in } \mathbb{R}, \tag{1}$$

$$\frac{\partial u}{\partial x}(0,t) = 0 \text{ for all } t \text{ in } \mathbb{R}, \tag{2}$$

$$u(L,t) = 0 \text{ for all } t \text{ in } \mathbb{R}, \tag{3}$$

$$u(x,0) = f(x) \text{ for } 0 \le x \le L, \text{ and} \tag{4}$$

$$\frac{\partial u}{\partial t}(x,0) = g(x) \text{ for } 0 \le x \le L. \tag{5}$$

Solution. Suppose that u is an elementary separated solution to (1). This means

$$u(x,t) = \varphi(x)h(t)$$

for some pair of one-place functions φ and h. Inserting this into (1), we have

$$\varphi(x)h''(t) = c^2\varphi''(x)h(t). \tag{6}$$

Assuming for now that

$$u(x.t) \neq 0,$$

and dividing each side of (6) by $\varphi(x)h(t)$, we have

$$\frac{\varphi(x)h''(t)}{\varphi(x)h(t)} = c^2\frac{\varphi''(x)h(t)}{\varphi(x)h(t)},$$

so

$$\frac{h''(t)}{h(t)} = c^2\frac{\varphi''(x)}{\varphi(x)}.$$

This holds for all t and all x with $0 \le x \le L$, so there is a constant K such that

$$\frac{h''(t)}{h(t)} = K = c^2\frac{\varphi''(x)}{\varphi(x)} \tag{7}$$

for all t and all x with $0 \le x \le L$. As a matter of notational convenience and so that we can more easily make use of our earlier work on two-point boundary value problems, we let

$$\lambda = -\frac{K}{c^2} \text{ so } K = -c^2\lambda.$$

From (7) we then have

$$-\varphi''(x) = \lambda\varphi(x) \text{ for all } x \text{ in } [0, L] \tag{8}$$

and

$$h''(t) = -\lambda c^2 h(t) \text{ for all } t. \tag{9}$$

It is worth noting that if

$$u(x,t) = \varphi(x)h(t)$$

and (8) and (9) hold, then

$$\begin{aligned} \frac{\partial^2 u}{\partial t^2}(x,t) &= \varphi(x)h''(t) = -\lambda c^2\varphi(x)h(t) \\ &= c^2\varphi''(x)h(t) = c^2\frac{\partial^2 u}{\partial x^2}(x,t) \end{aligned}$$

so the PDE (1)

$$\frac{\partial^2 u}{\partial t^2}(x,t) = c^2\frac{\partial^2 u}{\partial x^2}(x,t)$$

will be satisfied, and we no longer need to assume that $u(x,t) \neq 0$. Continuing with our assumption that

$$u(x,t) = \varphi(x)h(t)$$

We have from conditions (2) and (3)

$$\varphi'(0) = 0 \tag{10}$$

and

$$\varphi(L) = 0. \tag{11}$$

A proper listing of eigenvalues and eigenfunctions for (8), (10), and (11) is

$$\{\lambda_k\}_{k=1}^{\infty} \text{ and } \{\varphi_k\}_{k=1}^{\infty}$$

where

$$\lambda_k = (\frac{(2k-1)\pi}{2L})^2 \text{ for } k = 1, 2, \ldots$$

and

$$\varphi_k(x) = \cos\frac{(2k-1)\pi}{2L}x \text{ for all } x \text{ in } [0, L] \text{ and } k = 1, 2, \ldots.$$

The equation (9)

$$h''(t) = -c^2\lambda h(t)$$

is equivalent to

$$h''(t) + c^2\lambda h(t) = 0. \tag{12}$$

When $\lambda > 0$ as it must be because all eigenvalues for the problem (8), (10), and (11) are positive, a linearly independent pair of solutions to (12) is the pair whose values at t are

$$\cos\sqrt{\lambda}ct \text{ and } \sin\sqrt{\lambda}ct.$$

Thus when $\lambda = \lambda_k$ the solutions to (9) are linear combinations of the functions h_{1k} and h_{2k} where

$$h_{1k}(t) = \cos\sqrt{\lambda_k}ct \text{ and } h_{2k}(t) = \sin\sqrt{\lambda_k}ct.$$

We expect that the solution to the problem consisting of (1) through (5) will be of the form

$$u(x,t) = \sum_{k=1}^{\infty} \varphi_k(x)[A_k h_{1k}(t) + B_k h_{2k}(t)] \tag{13}$$

for some sequences of constants $\{A_k\}_{k=1}^{\infty}$ and $\{B_k\}_{k=1}^{\infty}$.

Condition (4)

$$u(x, 0) = f(x) \text{ for } x \text{ in } [0, L],$$

implies

$$f = \sum_{k=1}^{\infty} \varphi_k[A_k h_{1k}(0) + B_k h_{2k}(0)] = \sum_{k=1}^{\infty}[A_k \cos 0 + B_k \sin 0]\varphi_k = \sum_{k=1}^{\infty} A_k\varphi_k.$$

Since $\{\varphi_k\}_{k=1}^{\infty}$ is an orthogonal sequence of non zero functions this implies

$$A_k = \frac{< f, \varphi_k >}{< \varphi_k, \varphi_k >}$$

so for $k = 1, 2, \ldots$where the inner product is defined by

$$< \alpha, \beta >= \int_0^L \alpha(x)\beta(x)dx.$$

For this sequence $\{\varphi_k\}$,

$$< \varphi_k, \varphi_k >= \int_0^L (\cos\frac{(2k-1)\pi}{2L}x)^2 dx = \frac{L}{2} \text{ for } k = 1, 2, \ldots.$$

Returning to (13) we expect

$$\frac{\partial u}{\partial t}(x,t)=\sum_{k=1}^{\infty}\varphi_k(x)[A_k h_{1k}'(t)+B_k h_{2k}'(t)].$$

Condition (5)

$$\frac{\partial u}{\partial t}(x,0)=g(x) \text{ for all } x \text{ in } [0,L]$$

implies

$$g=\sum_{k=1}^{\infty}\varphi_k[A_k h_{1k}'(0)+B_k h_{2k}'(0)]=\sum_{k=1}^{\infty}(\frac{(2k-1)\pi}{2L}c)[-A_k\sin 0+B_k\cos 0]\varphi_k=\sum_{k=1}^{\infty}(\frac{(2k-1)\pi}{2L}c)B_k\varphi_k$$

so

$$(\frac{(2k-1)\pi}{2L}c)B_k=\frac{<g,\varphi_k>}{<\varphi_k,\varphi_k>} \text{ or } B_k=(\frac{2}{L})(\frac{2L}{(2k-1)\pi c})<g,\varphi_k>$$

for $k=1,2,3,\ldots$.In summary, the solution to the original problem (1) through (5) is u where

$$u(x,t)=\sum_{k=1}^{\infty}[A_k\cos\frac{(2k-1)\pi}{2L}ct+B_k\sin\frac{(2k-1)\pi}{2L}ct]\cos\frac{(2k-1)\pi x}{2L}$$

in which

$$A_k=\frac{2}{L}\int_0^L f(x)\cos\frac{(2k-1)\pi x}{2L}dx \text{ for } k=1,2,\ldots$$

and

$$B_k=\frac{4}{(2k-1)\pi c}\int_0^L g(x)\cos\frac{(2k-1)\pi x}{2L}dx \text{ for } k=1,2,\ldots$$

8. Suppose that

$$f(t)=3\sin 2t-3\cos 2t.$$

Find numbers $\mathcal{A}$, ω, and θ with $\mathcal{A}\geq 0$ and $0\leq\theta<2\pi$ such that

$$f(t)=\mathcal{A}\sin(\omega t+\theta)$$

Solution. (5 points for $\mathcal{A}$, 5 points for θ, and 0 points for ω) The way to solve this problem was shown in the note 'Standing Wave Solutions.'

$$\begin{aligned}\mathcal{A}\sin(\omega t+\theta) &= \mathcal{A}\sin\omega t\cos\theta+\mathcal{A}\cos\omega t\sin\theta\\ &= \mathcal{A}\sin\theta\cos\omega t+\mathcal{A}\cos\theta\sin\omega t\end{aligned}$$

This is

$$3\sin 2t-3\cos 2t=-3\cos 2t+3\sin 2t$$

if and only if $\omega=2$,

$$\mathcal{A}\sin\theta=-3 \text{ and } \mathcal{A}\cos\theta=3 \qquad (\#)$$

If these last two equations hold,

$$\mathcal{A}^2 \sin^2\theta = 9 \text{ and } \mathcal{A}^2 \cos^2\theta = 9$$

so

$$\mathcal{A}^2(\sin^2\theta + \cos^2\theta) = 18$$

so

$$\mathcal{A}^2 = 18$$

and since $\mathcal{A} \geq 0$,

$$\mathcal{A} = \sqrt{18} = 3\sqrt{2}.$$

From (#) we have

$$-1 = \frac{-3}{3} = \frac{\mathcal{A}\sin\theta}{\mathcal{A}\cos\theta} = \tan\theta.$$

Since $\mathcal{A}\sin\theta < 0$, $\mathcal{A}\cos\theta > 0$, and $0 \leq \theta < 2\pi$, it follows that

$$\theta = \frac{7\pi}{4}.$$