Examples of Expansions of Functions in Terms of an Orthogonal Sequence

If $\{\lambda_k\}_{k=k_0}^{\infty}$ and $\{\varphi_k\}_{k=k_0}^{\infty}$ is a proper listing of eigenvalues and eigenfunctions for a self-adjoint two-point boundary value problem over the interval $[a,b]$ and f is a function defined on $[a,b]$, the series for f determined by $\{\varphi_k\}_{k=k_0}^{\infty}$ is the sequence of functions $\{S_n\}_{n=k_0}^{\infty}$ given by

$$S_n(x) = \sum_{k=k_0}^{n} \frac{< f, \varphi_k >}{< \varphi_k, \varphi_k.} \varphi_k(x)$$

for all x in $[a,b]$ provided that $< f, \varphi_k >$ exists for each k.

A proper listing of eigenvalues and eigenfunctions for

$$\begin{aligned} -\varphi'' &= \lambda\varphi \text{ on } [0,L], \\ \varphi(0) &= 0, \text{ and} \\ \varphi(L) &= 0 \end{aligned}$$

is $\{\lambda_k\}_{k=0}^{\infty}$ and $\{\varphi_k\}_{k=0}^{\infty}$ where

$$\lambda_k = \left(\frac{k\pi}{L}\right)^2 \text{ and } \varphi_k(x) = \sin\frac{k\pi x}{L}.$$

Example 1. If

$$f(x) = 1 \text{ for } 0 \le x \le L$$

the series for f determined by this orthogonal sequence is $\{S_n\}_{n=1}^{\infty}$ where

$$S_n(x) = \sum_{k=1}^{n} \frac{< f, \varphi_k >}{< \varphi_k, \varphi_k >} \varphi_k(x) = \sum_{k=1}^{n} \frac{\int_0^L 1 \cdot \sin\frac{k\pi x}{L} dx}{\int_0^L \sin\frac{k\pi x}{L} \cdot \sin\frac{k\pi x}{L} dx} \sin\frac{k\pi x}{L}$$

$$\begin{aligned} \int_0^L 1 \cdot \sin\frac{k\pi x}{L} dx &= \int_0^L \sin\frac{k\pi x}{L} dx = -\frac{L}{k\pi}\left[\cos\frac{k\pi x}{L}\right]_{x=0}^{x=L} \\ &= -\frac{L}{k\pi}\left[\cos k\pi - \cos 0\right] = -\frac{L}{k\pi}\left[(-1)^k - 1\right] \\ &= \frac{L}{k\pi}\left[1 - (-1)^k\right] \end{aligned}$$

and

$$
\begin{aligned}
\int_0^L \sin\frac{k\pi x}{L}\cdot\sin\frac{k\pi x}{L}dx &= \int_0^L \sin^2\frac{k\pi x}{L}dx = \int_0^L \frac{1}{2}[1-\cos\frac{2k\pi x}{L}]dx \\
&= \left[\frac{x}{2}-\frac{L}{2k\pi}\sin\frac{2k\pi x}{L}\right]_{x=0}^{x=L} = \left[\frac{L}{2}-\frac{L}{2k\pi}\sin\frac{2k\pi L}{L}\right]-\left[\frac{0}{2}-\frac{L}{2k\pi}\sin\frac{2k\pi 0}{L}\right] \\
&= \left[\frac{L}{2}-\frac{L}{2k\pi}\sin 2k\pi\right] = \left[\frac{L}{2}-0\right] \\
&\frac{L}{2}
\end{aligned}
$$

Thus

$$
\frac{< f,\varphi_k >}{< \varphi_k,\varphi_k >} = \frac{\frac{L}{k\pi}\left[1-(-1)^k\right]}{\frac{L}{2}} = \frac{2}{k\pi}\left[1-(-1)^k\right]
$$

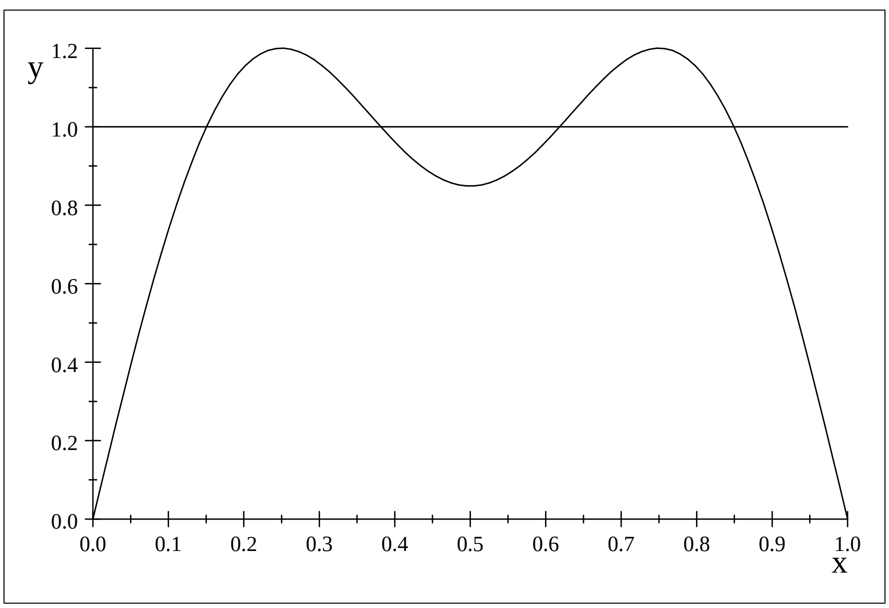
and

$$
S_n(x) = \sum_{k=1}^{n}\frac{2}{k\pi}\left[1-(-1)^k\right]\sin\frac{k\pi x}{L}
$$

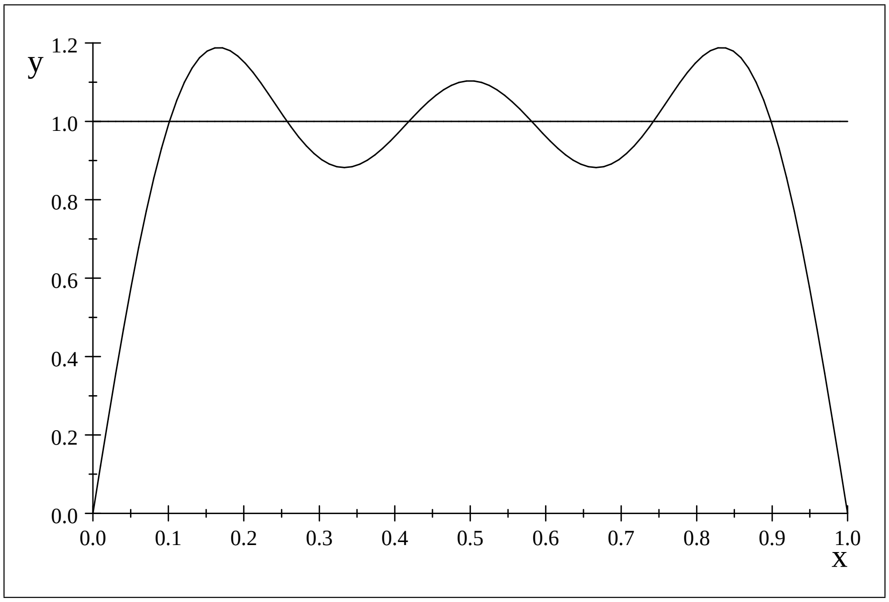
or

$$
S_n(x) = \frac{2}{\pi}\sum_{k=1}^{n}\frac{\left[1-(-1)^k\right]}{k}\sin\frac{k\pi x}{L}
$$

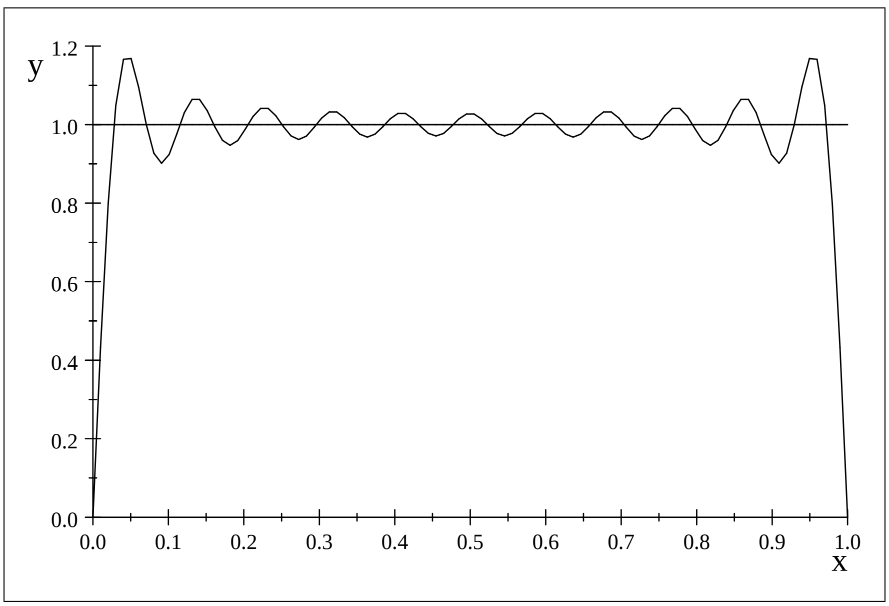
The following graphs are for the case where $L=1$.



f and S_3



f and S_5



f and S_{21}

Example 2. If

$$f(x) = x \text{ for } 0 \leq x \leq L$$

the series for f determined by this orthogonal sequence is $\{S_n\}_{n=1}^{\infty}$ where

$$S_n(x) = \sum_{k=1}^{n} \frac{< f, \varphi_k >}{< \varphi_k, \varphi_k >} \varphi_k(x) = \sum_{k=1}^{n} \frac{\int_0^L x \sin \frac{k\pi x}{L} dx}{\int_0^L \sin \frac{k\pi x}{L} \cdot \sin \frac{k\pi x}{L} dx} \sin \frac{k\pi x}{L}.$$

Using integration by parts,

$$\begin{aligned}
\int_0^L x \sin\frac{k\pi x}{L}dx &= \left[x\cdot(-\frac{L}{k\pi}\cos\frac{k\pi x}{L})\right]_{x=0}^{x=L} - \int_0^L 1\cdot(-\frac{L}{k\pi}\cos\frac{k\pi x}{L})dx \\
&= -\frac{L^2}{k\pi}\cos k\pi + \frac{L}{k\pi}\int_0^L \cos\frac{k\pi x}{L}dx \\
&= -\frac{L^2}{k\pi}\cos k\pi + (\frac{L}{k\pi})^2\left[\sin\frac{k\pi x}{L}\right]_{x=0}^{x=L} \\
&= \frac{(-1)^{k+1}L^2}{k\pi}
\end{aligned}$$

because $\cos k\pi = (-1)^k$ and $\sin k\pi = \sin 0 = 0$.

As in Example 1,

$$\int_0^L \sin\frac{k\pi x}{L}\cdot\sin\frac{k\pi x}{L}dx = \frac{L}{2}$$

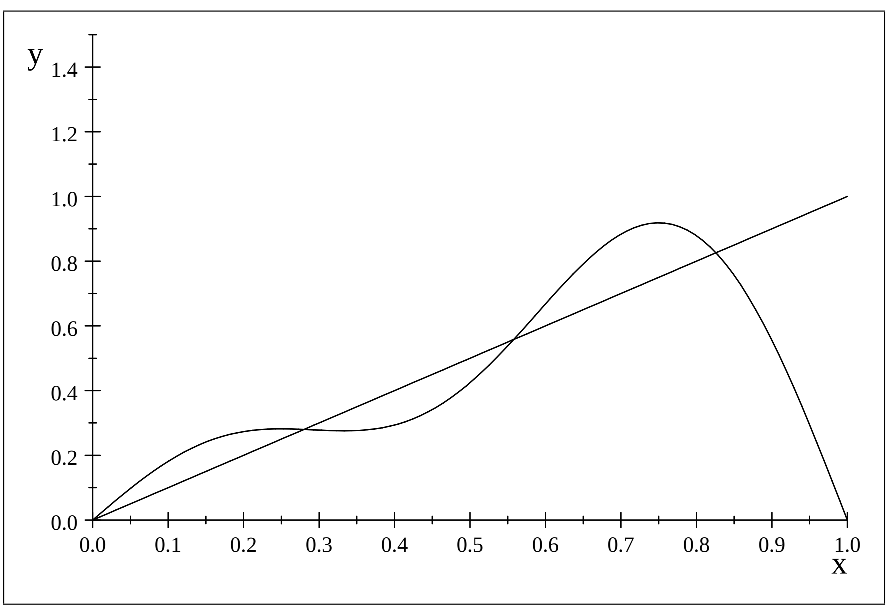
so

$$S_n(x) = \sum_{k=1}^{n}\frac{2(-1)^{k+1}L}{k\pi}\sin\frac{k\pi x}{L}$$

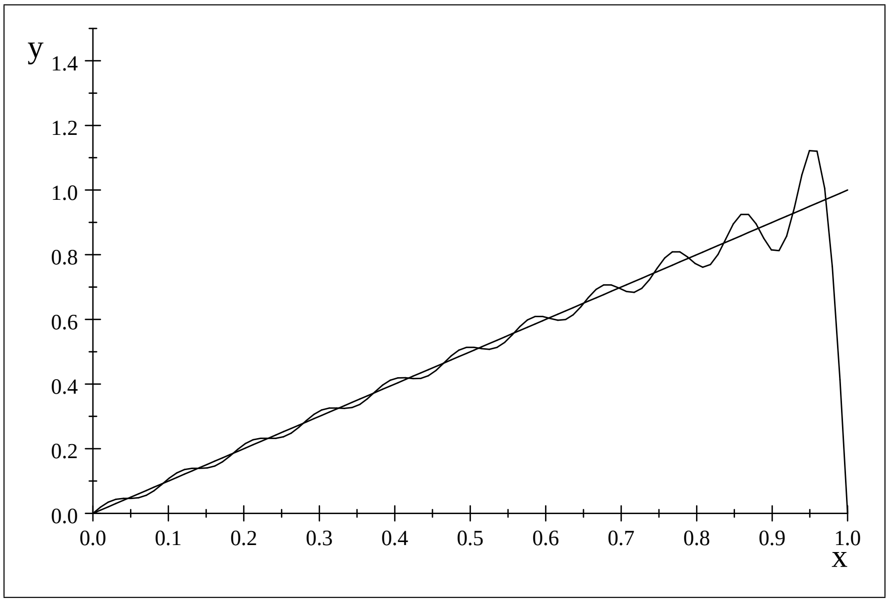
or

$$S_n(x) = \frac{2L}{\pi}\sum_{k=1}^{n}\frac{(-1)^{k+1}}{k}\sin\frac{k\pi x}{L}$$

The following graphs are for the case where $L = 1$.



f and S_3



f and S_{21}

Example 3. If f is given by

$$f(x) = x(L-x) \text{ for } 0 \leq x \leq L$$

the series for f determined by this orthogonal sequence is $\{S_n\}_{n=1}^{\infty}$ where

$$S_n(x) = \sum_{k=1}^{n} \frac{< f, \varphi_k >}{< \varphi_k, \varphi_k >} \varphi_k(x) = \sum_{k=1}^{n} \frac{\int_0^L x(L-x) \sin \frac{k\pi x}{L} dx}{\int_0^L \sin \frac{k\pi x}{L} \cdot \sin \frac{k\pi x}{L} dx} \sin \frac{k\pi x}{L}.$$

Integration by parts twice shows that

$$\int_0^L x(L-x) \sin \frac{k\pi x}{L} dx = \frac{2L^3}{\pi^3 k^3}(1-(-1)^k).$$

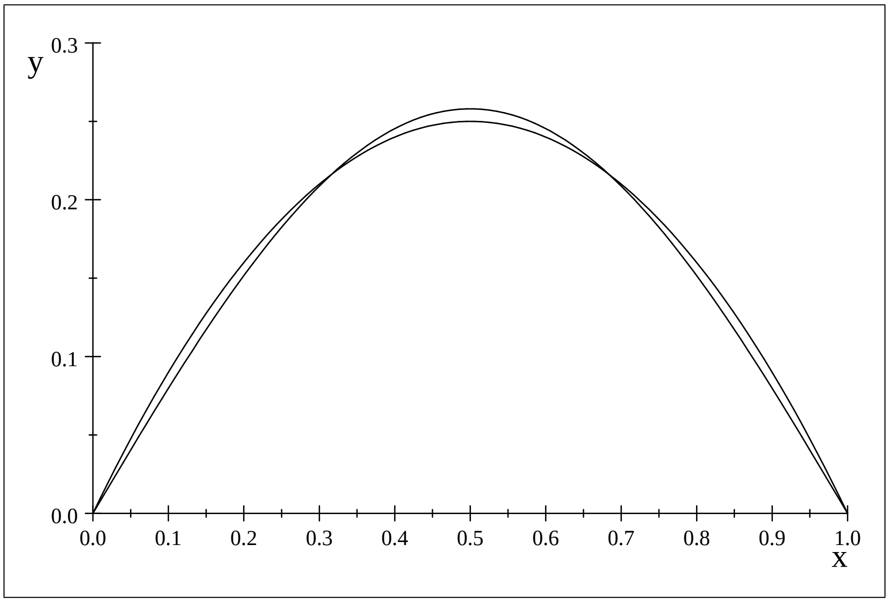
So

$$\begin{aligned} \frac{\int_0^L x(L-x) \sin \frac{k\pi x}{L} dx}{\int_0^L \sin \frac{k\pi x}{L} \cdot \sin \frac{k\pi x}{L} dx} &= \frac{\frac{2L^3}{\pi^3 k^3}(1-(-1)^k)}{\frac{L}{2}} \\ &= \frac{4L^2}{\pi^3 k^3}(1-(-1)^k) \end{aligned}$$

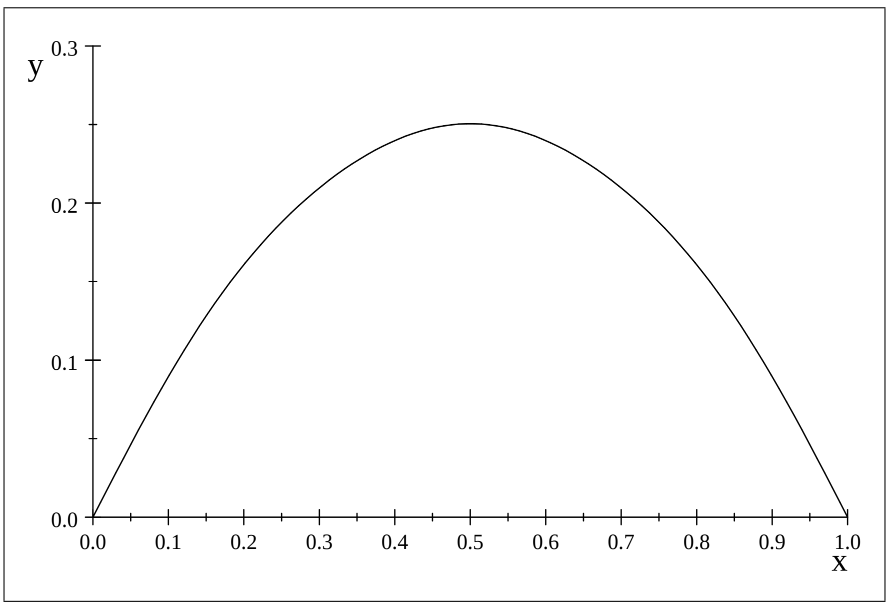
Thus

$$S_n(x) = \frac{4L^2}{\pi^3} \sum_{k=1}^{n} \frac{1}{k^3}(1-(-1)^k) \sin \frac{k\pi x}{L}.$$

The following graphs are for the case where $L = 1$.



f and S_1



f and S_5