Note. In Section 2.1 of the notes being given inclass, we are considering only a special case of the problem (1), (2), and (3) which you will find in the notes that follow. In class we have have taken $p(x) = 1$, $q(x) = 0$, and $w(x) = 1$ so that (1) becomes

$$-\varphi'' = \lambda\varphi.$$

We have taken $M_{11} = \beta_1$, $M_{12} = \beta_2$, $M_{21} = 0$, $M_{22} = 0$, $N_{11} = 0$, $N_{12} = 0$, $N_{21} = \beta_3$, and $N_{22} = \beta_4$ so that (2) becomes

$$\beta_1\varphi(a) + \beta_2\varphi'(a) = 0$$

and (3) becomes

$$\beta_3\varphi(b) + \beta_4\varphi'(b) = 0.$$

You can also find relevent information in Sections 5.3 and 5.5 of the text. Take $p(x) = \sigma(x) = 1$ and $q(x) = 0$ in Equation (5.3.1) on page 155.

Regular Two-Point Boundary Value Problems

Philip W. Walker

Suppose that a and b are real numbers with $a < b$, each of p, q, and w is a continuous real valued function with domain $[a, b]$, the function p has a continuous first derivative, each of $p(x)$ and $w(x)$ is positive for all x in $[a, b]$, each of M_{ij} and N_{ij} is a real number for $i = 1, 2$ and $j = 1, 2$, and the quadruples $(M_{11}, M_{12}, N_{11}, N_{12})$ and $(M_{21}, M_{22}, N_{21}, N_{22})$ are linearly independent. Suppose that the operator τ is given by

$$\tau\varphi = -(p\varphi')' - q\varphi$$

whenever φ is a twice differentiable real valued function with domain $[a, b]$.

We shall be concerned with finding the real numbers λ and real valued functions φ such that

$$\tau\varphi = \lambda w\varphi \text{ on } [a, b], \tag{1}$$

$$M_{11}\varphi(a) + M_{12}\varphi'(a) + N_{11}\varphi(b) + N_{12}\varphi'(b) = 0, \tag{2}$$

and

$$M_{21}\varphi(a) + M_{22}\varphi'(a) + N_{21}\varphi(b) + N_{22}\varphi'(b) = 0. \tag{3}$$

Remark 1. *The equation (1) is equivalent to*

$$p\varphi'' + p'\varphi' + (q + \lambda w)\varphi = 0.$$

Thus (1) is a regular second order linear homogeneous differential equation.

Remark 2. *The zero function on $[a, b]$ (i.e. the function φ such that $\varphi(x) = 0$ for all x in $[a, b]$) is always a solution to (1), (2), and (3).*

Definition. Saying that $\lambda_0$is an **eigenvalue** for the problem (1), (2), and (3) means that λ_0 is a real number and (1), (2), and (3) hold for some function φ other than the zero function when $\lambda = \lambda_0$. When λ_0 is an eigenvalue, saying that φ_0 is an **eigenfunction** corresponding to λ_0 means that φ_0 is a function other than the zero function and (1), (2), and (3) hold when $\lambda = \lambda_0$ and $\varphi = \varphi_0$. When λ_0 is an eigenvalue, the **eigenspace** corresponding to λ_0 consists of all functions φ satisfying (1), (2), and (3).

Remark 3. *Suppose that λ_0 is an eigenvalue for (1), (2), and (3). Since the eigenspace corresponding to λ_0 is a subspace of the set of all solutions to (1), the eigenspace is either one-dimensional or two-dimensional.*

Example 1. Consider the problem (L is a positive number)

$$-\varphi'' = \lambda\varphi \text{ on } [0, L], \tag{4}$$

$$\varphi(0) = 0, \tag{5}$$

and

$$\varphi(L) = 0. \tag{6}$$

In order to find the eigenvalues and eigenfunctions we will consider three cases.

Case 1: Suppose that $\lambda < 0$. Then φ satisfies (4) only in case

$$\varphi(x) = c_1 \cosh \sqrt{-\lambda}x + c_2 \sinh \sqrt{-\lambda}x$$

for some pair of numbers c_1 and c_2 and all x in $[0, L]$. Since $\cosh(0) = 1$ and $\sinh(0) = 0$,it follows that (5) will also hold only in case

$$c_1 = 0.$$

Thus (4) and (5) hold only in case

$$\varphi(x) = c_2 \sinh \sqrt{-\lambda}x \tag{7}$$

for some number c_2. Since $\sinh z = 0$ only in case $z = 0$ and $\sqrt{-\lambda}L > 0$, it follows that (4), (5), and (6) hold only in case (7) holds with $c_2 = 0$ or

$$\varphi(x) = 0 \text{ for all } x \text{ in } [0, L].$$

Thus there are no negative eigenvalues.

Case 2: Suppose that $\lambda = 0$. Then φ satisfies (4) only in case

$$\varphi(x) = c_1 + c_2 x \tag{8}$$

for some pair of numbers c_1 and c_2 and all x in $[0, L]$. Thus it follows that (4), (5), and (6) hold only in case (8) holds with $c_1 = c_2 = 0$ or

$$\varphi(x) = 0 \text{ for all } x \text{ in } [0, L].$$

Thus zero is not an eigenvalue.

Case 3: Suppose that $\lambda > 0$. Then φ satisfies (4) only in case

$$\varphi(x) = c_1 \cos \sqrt{\lambda}x + c_2 \sin \sqrt{\lambda}x$$

for some pair of numbers c_1 and c_2 and all x in $[0, L]$. Since $\cos(0) = 1$ and $\sin(0) = 0$,it follows that (5) will also hold only in case

$$c_1 = 0.$$

Thus (4) and (5) hold only in case

$$\varphi(x) = c_2 \sin \sqrt{\lambda} x \tag{9}$$

for some number c_2. Since $\sin z = 0$ only in case z is an integral multiple of π and $\lambda > 0$, it follows that (4), (5), and (6) hold with φ different from the zero function only in case (9) holds with $c_2 \neq 0$ and $\sqrt{\lambda} L = k\pi$ or $\sqrt{\lambda} = k\pi\ /L$ or

$$\lambda = \left(\frac{k\pi}{L}\right)^2 \text{ for some positive integer } k. \tag{10}$$

As we will see below, all eigenvalues of (4), (5), and (6) must be real numbers. Thus λ is an eigenvalue only in case (10) holds, and when (10) holds, φ is a corresponding eigenfunction only in case

$$\varphi(x) = c \sin \frac{k\pi x}{L} \text{ for some number } c \neq 0 \text{ and all } x \text{ in } [0, L].$$

From this, it follows that each eigenspace is one dimensional.

Remark 4. *Here is a procedure for finding the eigenvalues and eigenfunctions of the problem consisting of (1), (2), and (3). For each complex number λ, let (u_λ, v_λ) be a linearly independent pair of solution to (1). Then φ satisfies (1) only in case*

$$\varphi(x) = c_1 u_\lambda(x) + c_2 v_\lambda(x) \textit{ for all } x \textit{ in } [a, b]$$

for some pair of complex numbers (c_1, c_2). Moreover, φ is different from the zero function only in case at least one of c_1 and c_2 is different from zero. When $\varphi = c_1 u_\lambda + c_2 v_\lambda$ then

$$\varphi'(x) = c_1 u'_\lambda(x) + c_2 v'_\lambda(x) \textit{ for all } x \textit{ in } [a, b]$$

so

$$\begin{bmatrix} \varphi(x) \\ \varphi'(x) \end{bmatrix} = \Phi_\lambda(x) \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \textit{ for all } x \textit{ in } [a, b]$$

where

$$\Phi_\lambda(x) = \begin{bmatrix} u_\lambda(x) & v_\lambda(x) \\ u'_\lambda(x) & v'_\lambda(x) \end{bmatrix}.$$

Conditions (2) and (3) together are equivalent to

$$M \begin{bmatrix} \varphi(a) \\ \varphi'(a) \end{bmatrix} + N \begin{bmatrix} \varphi(b) \\ \varphi'(b) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

where

$$M = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \tag{11}$$

and

$$N = \begin{bmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{bmatrix}. \tag{12}$$

Thus φ will satisfy (1), (2), (3) only in case

$$\varphi = c_1 u_\lambda + c_2 v_\lambda$$

and

$$M\Phi_\lambda(a) \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} + N\Phi_\lambda(b) \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

or

$$D(\lambda)\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

where

$$D(\lambda) = M\Phi_\lambda(a) + N\Phi_\lambda(b).$$

Moreover, there will be a solution where at least one of c_1 and c_2 is not zero only in case

$$\Delta(\lambda) = 0$$

where

$$\Delta(\lambda) = \det D(\lambda).$$

We have established the following.

Theorem 1. *The eigenvalues of the problem (1), (2), and (3) are the zeros of the function Δ, and if $\Delta(\lambda_0) = 0$ then φ is an eigenfunction corresponding to the eigenvalue λ_0 only in case*

$$\varphi = c_1 u_{\lambda_0} + c_2 v_{\lambda_0}$$

where and at least one of the numbers c_1 and c_2 is different from zero and

$$D(\lambda_0)\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

From this it follows that if λ_0 is an eigenvalue, the corresponding eigenspace is two-dimensional when

$$D(\lambda_0) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

and is one-dimensional when

$$D(\lambda_0) \neq \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Definition. The statement that the problem (1), (2), and (3) is **self-adjoint** means that

$$\int_a^b (\tau f)g = \int_a^b f\tau g$$

whenever each of f and g is a twice continuously differentiable function with domain $[a, b]$ and each of f and g satisfies the conditions (2) and (3).

Example 2. Consider the problem (4), (5), and (6) given in the last example. Here

$$\tau\varphi = -\varphi''.$$

So if each of f and g has a continuous second derivative and each satisfies (5) and (6),

$$\begin{aligned} \int_0^L (\tau f)g &= \int_0^L -f''g = -[f'g]_0^L + \int_0^L f'g' = 0 + \int_0^L f'g' \\ &= [fg']_0^L + \int_0^L f(-g'') = \int_0^L f(-g'') = \int_0^L f\tau g \end{aligned}$$

Thus the problem (4), (5), and (6) is self-adjoint.

The following theorem gives a straightforward test for self-adjointness.

Theorem 2. *The problem (1), (2), and (3) is self-adjoint only in case*

$$p(b)M \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} M^T = p(a)N \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} N^T.$$

where M and N are given by (11) and (12).

Remember that w is a continuous positive valued function defined on $[a, b]$.

Whenever each of f and g is a piecewise continuous real-valued function defined on $[a, b]$, the **inner product** of f and g is denoted by $< f, g >$ and is defined by

$$< f, g >= \int_a^b f(x)g(x)w(x)dx.$$

Suppose that each of f and g is a piecewise continuous function defined on $[a, b]$. The statement that f and g are **orthogonal** means that

$$< f, g >= 0.$$

Suppose that $\varphi_{k_0}, \varphi_{k_0+1}, \varphi_{k_0+2}, \ldots$ is a sequence of piecewise continuous functions defined on $[a.b]$. The statement that $\varphi_{k_0}, \varphi_{k_0+1}, \varphi_{k_0+2}, \ldots$ is orthogonal means that

$$< \varphi_i, \varphi_j >= 0$$

whenever $i \neq j$.

Theorem 3. *If the problem (1), (2), and (3) is self-adjoint, then all eigenvalues are real and eigenfunctions corresponding to different eigenvalues are orthogonal.*

The proof is given in case complex numbers and complex valued functions are allowed. The bar means complex conjugate and the inner product is given by

$$< u, v >= \int_a^b u(x)\overline{v(x)}w(x)dx$$

Proof. Suppose that λ is an eigenvalue and φ is a corresponding eigenfunction. Then

$$\begin{aligned} \lambda < \varphi, \varphi > &= < \lambda\varphi, \varphi >= \int_a^b \lambda w \varphi \overline{\varphi} = \int_a^b (\tau\varphi)\overline{\varphi} = \int_a^b \varphi \overline{\tau\varphi} \\ &= \int_a^b \varphi \overline{\lambda w \varphi} = \overline{\lambda} \int_a^b \varphi\overline{\varphi} w = \overline{\lambda} < \varphi, \varphi > . \end{aligned}$$

Since φ is continuous and not the zero function it follows that $< \varphi, \varphi > \neq 0$. Thus $\lambda = \overline{\lambda}$ showing that λ is real. Suppose now that λ and μ are eigenvalues, $\lambda \neq \mu$, φ is an eigenfunction corresponding to λ, and ψ is an eigenfunction corresponding to μ.

$$\begin{aligned} \lambda < \varphi, \psi > &= < \lambda\varphi, \psi >= \int_a^b \lambda w \varphi \overline{\psi} = \int_a^b (\tau\varphi)\overline{\psi} \\ &= \int_a^b \varphi \overline{\tau\psi} = \int_a^b \varphi \overline{\mu w \psi} = \int_a^b \varphi \mu w \overline{\psi} = \mu < \varphi, \psi > \end{aligned}$$

Thus

$$(\lambda - \mu) < \varphi, \psi >= 0,$$

and since $\lambda \neq \mu$, it follows that $< \varphi, \psi >= 0$. $\square$

Theorem 4. *Suppose that the problem (1), (2), and (3) is self-adjoint. There will be infinitely many eigenvalues and they can be arranged in a nondecreasing sequence* $\lambda_{k_0}, \lambda_{k_0+1}, \lambda_{k_0+2}, \ldots$ *with*

$$\lim_{k\to\infty} \lambda_k = \infty.$$

When necessary, the Gram-Schmidt process can be used to convert a linearly independent sequence into an orthogonal one. The following theorem gives the process for a linearly independent pair.

Theorem 5. *Suppose that* λ_0 *is an eigenvalue and the corresponding eigenspace is two-dimensional. An orthogonal basis for this eigenspace is* (α, β) *where*

$$\alpha = u_{\lambda_0}$$

and

$$\beta = v_{\lambda_0} - \frac{< v_{\lambda_0}, \alpha >}{< \alpha, \alpha >}\alpha$$

Suppose that the problem (1), (2), and (3) is self-adjoint. A **proper listing of eigenvalues and eigenfunctions** for the problem consists of a nondecreasing sequence of eigenvalues $\lambda_{k_0}, \lambda_{k_0+1}, \lambda_{k_0+2}, \ldots$ in which each eigenvalue is listed exactly the number of times that is the dimension of the corresponding eigenspace and an orthogonal sequence of eigenfunctions $\varphi_{k_0}, \varphi_{k_0+1}, \varphi_{k_0+2}, \ldots$ in which φ_j is an eigenfunction corresponding to$\lambda_j$for $j = k_0, k_0+1, k_0+2, \ldots$.

Theorem 6. *If the problem (1), (2), and (3) is self-adjoint, then there is a proper listing of eigenvalues and eigenfunctions for the problem.*

The statement that the problem (1), (2), and (3) is a **Sturm-Liouville** problem means that the conditions (2) and (3) are equivalent to ones of the form

$$\begin{aligned} M_{11}\varphi(a) + M_{12}\varphi'(a) &= 0 \text{ and} \\ N_{21}\varphi(b) + N_{22}\varphi'(b) &= 0 \end{aligned}$$

where each of M_{11}, M_{12}, N_{21}, and N_{22} is real, at least one of M_{11} and M_{12} is not zero, and at least one on N_{21} and N_{22} is not zero.

Theorem 7. *All Sturm-Liouville problems are self-adjoint and have eigenspaces that are all one-dimensional.*

Suppose that the problem (1), (2), and (3) is self-adjoint and $\{\lambda_k\}_{k=k_0}^{\infty}$ and $\{\varphi_k\}_{k=k_0}^{\infty}$ is a proper listing of eigenvalues and eigenfunctions. When f is a function that is piecewise continuous on $[a, b]$, **the series for f determined by** $\{\varphi_k\}_{k=k_0}^{\infty}$ is the sequence of functions $\{S_n\}_{n=k_0}^{\infty}$ given by

$$S_n(x) = \sum_{k=k_0}^{n} \frac{< f, \varphi_k >}{< \varphi_k, \varphi_k >}\varphi_k(x) \text{ for all } x \text{ in } [a, b] \text{ and } n = k_0, k_0+1, k_0+2, \ldots.$$

Theorem 8. *Suppose that the problem (1), (2), and (3) is self-adjoint and* $\{\lambda_k\}_{k=k_0}^{\infty}$ *and* $\{\varphi_k\}_{k=k_0}^{\infty}$ *is a proper listing of eigenfunctions and eigenvalues, and suppose that* f *is a function that is piecewise continuous on* $[a.b]$*. (i) It follows that* $\{c_k\}_{k=k_0}^{\infty}$ *is a sequence of numbers and*

$$f = \sum_{k=k_0}^{\infty} c_k \varphi_k$$

with convergence in the mean (i.e.

$$\lim_{n\to\infty} < f - \sum_{k=k_0}^{n} c_k\varphi_k, f - \sum_{k=k_0}^{n} c_k\varphi_k >= 0 \text{)}$$

only in case

$$c_k = \frac{< f, \varphi_k >}{< \varphi_k, \varphi_k >} \text{ for } k = k_0, k_0+1, k_0+2, \ldots.$$

(ii) If f *is piecewise smooth and*

$$c_k = \frac{< f, \varphi_k >}{< \varphi_k, \varphi_k >} \text{ for } k = k_0, k_0+1, k_0+2, \ldots.$$

then

$$\lim_{n\to\infty} \sum_{k=k_0}^{n} \frac{< f, \varphi_k >}{< \varphi_k, \varphi_k >}\varphi_k(x) = \frac{1}{2}[f(x+) + f(x-)]$$

for each x *with* $a < x < b$*. (iii) If* f *has a continuous second derivative, satisfies the boundary conditions (2) and (3), and*

$$S_n(x) = \sum_{k=k_0}^{n} \frac{< f, \varphi_k >}{< \varphi_k, \varphi_k >}\varphi_k(x) \text{ for all } x \text{ in } [a,b] \text{ and } n = k_0, k_0+1, k_0+2, \ldots,$$

then $\{S_n\}$ *converges uniformly to* f *on* $[a,b]$*.*

Theorem 9. *(The Rayleigh Quotient) Suppose that the problem (1), (2), and (3) is self-adjoint, that* λ *is an eigenvalue, and that* φ *is a corresponding eigenfunction. It follows that*

$$\lambda = \frac{[p(a)\varphi'(a)\varphi(a) - p(b)\varphi'(b)\varphi(b)] + \int_a^b (p(\varphi')^2 - q\varphi|^2)}{\int_a^b (\varphi^2 w)}.$$

Thus if $q(x) \leq 0$ *for all* x *in* $[a,b]$ *and the boundary conditions (2) and (3) imply* $[p(a)\varphi'(a)\varphi(a) - p(b)\varphi'(b)\varphi(b)] \geq 0$*, it follows that* $\lambda \geq 0$*. If it is also true that the non-zero constant functions fail to satisfy either (2) or (3) then*

$$\int_a^b (p\varphi'^2) > 0$$

and it follows that $\lambda > 0$*.*

Remark 5. *In the special case where* τ *is given by*

$$\tau\varphi = -\varphi'',$$

$w(x) = 1$*, the problem (1), (2), and (3) is self-adjoint,* λ *is an eigenvalue, and* φ *is a corresponding real valued eigenfunction, the Rayleigh Quotient becomes*

$$\lambda = \frac{[\varphi'(a)\varphi(a) - \varphi'(b)\varphi(b)] + \int_a^b (\varphi')^2}{\int_a^b (\varphi)^2}.$$

Remark 6. *In the special case where τ is given by*

$$\tau\varphi = -\varphi''$$

and $w(x) = 1$, equation (1) is equivalent to

$$\varphi'' + \lambda\varphi = 0,$$

and we will let the linearly independent pair of solutions (u_λ, v_λ) be given by

$$u_\lambda(x) = \begin{cases} \cosh\sqrt{-\lambda}x & \text{when} \quad \lambda < 0 \\ 1 & \text{when} \quad \lambda = 0 \\ \cos\sqrt{\lambda}x & \text{when} \quad \lambda > 0 \end{cases}$$

and

$$v_\lambda(x) = \begin{cases} \sinh\sqrt{-\lambda}x & \text{when} \quad \lambda < 0 \\ x & \text{when} \quad \lambda = 0 \\ \sin\sqrt{\lambda}x & \text{when} \quad \lambda > 0 \end{cases}.$$

With this definition of (u_λ, v_λ), note that

$$\Phi_\lambda(x) = \begin{bmatrix} \cosh\sqrt{-\lambda}x & \sinh\sqrt{-\lambda}x \\ \sqrt{-\lambda}\sinh\sqrt{-\lambda}x & \sqrt{-\lambda}\cosh\sqrt{-\lambda}x \end{bmatrix} \text{ when } \lambda < 0,$$

$$\Phi_\lambda(x) = \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} \text{ when } \lambda = 0, \text{ and}$$

$$\Phi_\lambda(x) = \begin{bmatrix} \cos\sqrt{\lambda}x & \sin\sqrt{\lambda}x \\ -\sqrt{\lambda}\sin\sqrt{\lambda}x & \sqrt{\lambda}\cos\sqrt{\lambda}x \end{bmatrix} \text{ when } \lambda > 0.$$

Example 3. Consider the problem

$$\begin{aligned} -\varphi'' &= \lambda\varphi \text{ on } [0, L], \\ \varphi(0) &= 0, \text{ and} \\ \varphi(L) &= 0. \end{aligned}$$

The boundary conditionsare equivalent to

$$\begin{aligned} 1 \cdot \varphi(0) + 0 \cdot \varphi'(0) + 0 \cdot \varphi(L) + 0 \cdot \varphi'(L) &= 0 \text{ and} \\ 0 \cdot \varphi(0) + 0 \cdot \varphi'(0) + 1 \cdot \varphi(L) + 0 \cdot \varphi'(L) &= 0 \end{aligned}$$

so they are equivalent to

$$M \begin{bmatrix} \varphi(0) \\ \varphi'(0) \end{bmatrix} + N \begin{bmatrix} \varphi(L) \\ \varphi'(L) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

where

$$M = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \text{ and } N = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

This is a Sturm-Liouville problem, so it is self-adjoint. If λ is an eigenvalue and φ is a corresponding real valued eigenfunction then (Rayleigh Quotient)

$$\begin{aligned} \lambda &= \frac{[\varphi'(0) \cdot 0 - \varphi'(L) \cdot 0)] + \int_0^L (\varphi')^2}{\int_0^L (\varphi)^2}, \text{ so} \\ \lambda &= \frac{\int_0^L (\varphi')^2}{\int_0^L (\varphi)^2}. \end{aligned}$$

Thus all eigenvalues are nonnegative. The nonzero constant functions do not satisfy the boundary conditions, so all eigenvalues are positive.

The boundary conditions are equivalent to

$$M\begin{bmatrix}\varphi(0)\\ \varphi'(0)\end{bmatrix}+N\begin{bmatrix}\varphi(L)\\ \varphi'(L)\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}$$

where

$$M=\begin{bmatrix}1&0\\0&0\end{bmatrix}\text{ and }N=\begin{bmatrix}0&0\\1&0\end{bmatrix}.$$

When $\lambda>0$,

$$D(\lambda)=\begin{bmatrix}1&0\\0&0\end{bmatrix}\begin{bmatrix}1&0\\0&\sqrt{\lambda}\end{bmatrix}+\begin{bmatrix}0&0\\1&0\end{bmatrix}\begin{bmatrix}\cos\sqrt{\lambda}L&\sin\sqrt{\lambda}L\\ -\sqrt{\lambda}\sin\sqrt{\lambda}L&\sqrt{\lambda}\cos\sqrt{\lambda}L\end{bmatrix}.$$

So

$$D(\lambda)=\begin{bmatrix}1&0\\ \cos\sqrt{\lambda}L&\sin\sqrt{\lambda}L\end{bmatrix}$$

and

$$\Delta(\lambda)=\det D(\lambda)=\sin\sqrt{\lambda}L.$$

From this we see that λ is an eigenvalue only in case

$$\sqrt{\lambda}L=k\pi\text{ or }\lambda=\left(\frac{k\pi}{L}\right)^2\text{ for some positive integer }k.$$

Note that

$$D(\left(\frac{k\pi}{L}\right)^2)=\begin{bmatrix}1&0\\(-1)^k&0\end{bmatrix}$$

so

$$D(\left(\frac{k\pi}{L}\right)^2)\begin{bmatrix}c_1\\c_2\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}$$

only in case

$$c_1=0.$$

From this it follows that φ is an eigenfunction corresponding to the eigenvalue $\left(\frac{k\pi}{L}\right)^2$ only in case

$$\varphi(x)=c_2\sin\frac{k\pi x}{L}$$

for all x in $[0,L]$ and some number $c_2\neq 0$.

Based on these observations, it follows that a proper listing of eigenvalues and eigenfunctions for this problem is $\{\lambda_k\}_{k=1}^{\infty}$ and $\{\varphi_k\}_{k=1}^{\infty}$ where

$$\lambda_k=\left(\frac{k\pi}{L}\right)^2\text{ for }k=1,2,3,\ldots\text{ and }\varphi_k(x)=\sin\frac{k\pi x}{L}\text{ for }k=1,2,3,\ldots\text{ and }0\le x\le L.$$

Computation shows that

$$<\varphi_k,\varphi_k>=\int_0^L(\varphi_k)^2=\frac{L}{2}\text{ for }k=1,2,3,\ldots.$$

Example 4. Consider the problem

$$\begin{aligned}-\varphi''&=\lambda\varphi\text{ on }[0,L],\\ \varphi'(0)&=0,\text{ and}\\ \varphi'(L)&=0.\end{aligned}$$

This is a Sturm-Liouville problem, so it is self-adjoint. If λ is an eigenvalue and φ is a corresponding real valued eigenfunction then (Rayleigh Quotient)

$$\begin{aligned}\lambda &= \frac{[0 \cdot \varphi(0) - 0 \cdot \varphi(L)] + \int_0^L (\varphi')^2}{\int_0^L (\varphi)^2}, \text{ so} \\ \lambda &= \frac{\int_0^L (\varphi')^2}{\int_0^L (\varphi)^2}.\end{aligned}$$

Thus all eigenvalues are nonnegative. The nonzero constant functions do satisfy the boundary conditions, so all that we can conclude at this point is that all eigenvalues are nonnegative.

The boundary conditions are equivalent to

$$M \begin{bmatrix} \varphi(0) \\ \varphi'(0) \end{bmatrix} + N \begin{bmatrix} \varphi(L) \\ \varphi'(L) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

where

$$M = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \text{ and } N = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

When $\lambda > 0$,

$$D(\lambda) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{\lambda} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\sqrt{\lambda}L & \sin\sqrt{\lambda}L \\ -\sqrt{\lambda}\sin\sqrt{\lambda}L & \sqrt{\lambda}\cos\sqrt{\lambda}L \end{bmatrix}.$$

So

$$D(\lambda) = \begin{bmatrix} 0 & \sqrt{\lambda} \\ -\sqrt{\lambda}\sin\sqrt{\lambda}L & \sqrt{\lambda}\cos\sqrt{\lambda}L \end{bmatrix}$$

and

$$\Delta(\lambda) = \det D(\lambda) = \lambda \sin\sqrt{\lambda}L.$$

From this we see that λ is a positive eigenvalue only in case

$$\sqrt{\lambda}L = k\pi \text{ or } \lambda = \left(\frac{k\pi}{L}\right)^2 \text{ for some positive integer } k.$$

Note that when k is a positive integer,

$$D(\left(\frac{k\pi}{L}\right)^2) = \begin{bmatrix} 0 & \frac{k\pi}{L} \\ 0 & \frac{k\pi}{L}(-1)^k \end{bmatrix},$$

so

$$D(\left(\frac{k\pi}{L}\right)^2) \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

only in case

$$c_2 = 0.$$

From this it follows that φ is an eigenfunction corresponding to the eigenvalue $\left(\frac{k\pi}{L}\right)^2$ when k is a positive integer only in case

$$\varphi(x) = c \cos\frac{k\pi x}{L}$$

for some number $c \neq 0$ and all x in $[0, L]$.

When $\lambda = 0$,

$$D(\lambda) = D(0) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & L \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}.$$

and

$$\Delta(\lambda) = \Delta(0) = 0.$$

Thus zero is an eigenvalue. Note that

$$D(0) \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

only in case $c_2 = 0$,so φ is an eigenfunction corresponding to the eigenvalue zero only in case

$$\varphi(x) = c$$

for some number $c \neq 0$ and all x in $[0, L]$.

Based on these observations, it follows that a proper listing of eigenvalues and eigenfunctions for this problem is $\{\lambda_k\}_{k=0}^{\infty}$ and $\{\varphi_k\}_{k=0}^{\infty}$ where

$$\lambda_0 = 0,\ \varphi_0\,(x) = 1 \text{ for } 0 \leq x \leq L,$$

$$\lambda_k = \left(\frac{k\pi}{L}\right)^2 \text{ for } k = 1, 2, 3, \ldots, \text{ and } \varphi_k(x) = \cos\frac{k\pi x}{L} \text{ for } k = 1, 2, 3, \ldots \text{ and } 0 \leq x \leq L.$$

Computation shows that

$$< \varphi_0, \varphi_0 >= \int_0^L (\varphi_0)^2 = L, \text{ and}$$

$$< \varphi_k, \varphi_k >= \int_0^L (\varphi_k)^2 = \frac{L}{2} \text{ for } k = 1, 2, 3, \ldots.$$

Example 5. Consider the problem

$$\begin{aligned} -\varphi'' &= \lambda\varphi \text{ on } [-L, L], \\ \varphi(-L) &= \varphi(L), \text{ and} \\ \varphi'(-L) &= \varphi'(L). \end{aligned}$$

This is a not a Sturm-Liouville problem; however it is self-adjoint. If λ is an eigenvalue and φ is a corresponding real valued eigenfunction then (Rayleigh Quotient)

$$\begin{aligned} \lambda &= \frac{[\varphi'(-L)\cdot\varphi(-L) - \varphi'(L)\cdot\varphi(L)] + \int_{-L}^{L}(\varphi')^2}{\int_{-L}^{L}(\varphi)^2}, \text{ so} \\ \lambda &= \frac{[\varphi'(L)\cdot\varphi(L) - \varphi'(L)\cdot\varphi(L)] + \int_{-L}^{L}(\varphi')^2}{\int_{-L}^{L}(\varphi)^2}, \text{so} \\ \lambda &= \frac{\int_{-L}^{L}(\varphi')^2}{\int_{-L}^{L}(\varphi)^2}. \end{aligned}$$

Thus all eigenvalues are nonnegative. The nonzero constant functions do satisfy the boundary conditions, so all that we can conclude at this point is that all eigenvalues are nonnegative.

The boundary conditions are equivalent to

$$M\begin{bmatrix}\varphi(-L)\\ \varphi'(-L)\end{bmatrix}+N\begin{bmatrix}\varphi(L)\\ \varphi'(L)\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}$$

where

$$M=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\text{ and }N=\begin{bmatrix}-1 & 0\\ 0 & -1\end{bmatrix}.$$

When $\lambda>0$,

$$D(\lambda)=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}\cos\sqrt{\lambda}L & -\sin\sqrt{\lambda}L\\ \sqrt{\lambda}\sin\sqrt{\lambda}L & \sqrt{\lambda}\cos\sqrt{\lambda}L\end{bmatrix}+\begin{bmatrix}-1 & 0\\ 0 & -1\end{bmatrix}\begin{bmatrix}\cos\sqrt{\lambda}L & \sin\sqrt{\lambda}L\\ -\sqrt{\lambda}\sin\sqrt{\lambda}L & \sqrt{\lambda}\cos\sqrt{\lambda}L\end{bmatrix}.$$

So

$$D(\lambda)=\begin{bmatrix}0 & -2\sin\sqrt{\lambda}L\\ 2\sqrt{\lambda}\sin\sqrt{\lambda}L & 0\end{bmatrix}$$

and

$$\Delta(\lambda)=\det D(\lambda)=4\sqrt{\lambda}\sin^2\sqrt{\lambda}L.$$

From this we see that λ is a positive eigenvalue only in case

$$\sqrt{\lambda}L=k\pi\text{ or }\lambda=\left(\frac{k\pi}{L}\right)^2\text{ for some positive integer }k.$$

When k is a positive integer,

$$D(\left(\frac{k\pi}{L}\right)^2)=\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}$$

so the eigenspace corresponding to $\left(\frac{k\pi}{L}\right)^2$ is two dimensional and a corresponding linearly independent pair of eigenfunctions is (u,v) where

$$u(x)=\cos\frac{k\pi x}{L}\text{ and }v(x)=\sin\frac{k\pi x}{L}$$

Computation shows that $<u,v>=0$; this pair is already orthogonal.

When $\lambda=0$,

$$D(\lambda)=D(0)=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}1 & -L\\ 0 & 1\end{bmatrix}+\begin{bmatrix}-1 & 0\\ 0 & -1\end{bmatrix}\begin{bmatrix}1 & L\\ 0 & 1\end{bmatrix}=\begin{bmatrix}0 & -2L\\ 0 & 0\end{bmatrix}.$$

and

$$\Delta(\lambda)=\Delta(0)=0.$$

Thus zero is an eigenvalue. Note that

$$D(0)\begin{bmatrix}c_1\\ c_2\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}$$

only in case $c_2=0$, so φ is an eigenfunction corresponding to the eigenvalue zero only in case

$$\varphi(x)=c$$

for some number $c\neq 0$ and all x in $[-L,L]$.

Thus a proper listing of eigenvalues and eigenfunctions for this problem is $\{\lambda_k\}_{k=0}^{\infty}$ and $\{\varphi_k\}_{k=0}^{\infty}$ where

$$\lambda_0=0,\ \varphi_0(x)=1\text{ for }-L\leq x\leq L,$$

$$\begin{aligned}\lambda_{2k-1} &= \lambda_{2k} = \left(\frac{k\pi}{L}\right)^2 \text{ for } k = 1,2,3,\ldots, \\ \varphi_{2k-1}(x) &= \cos\frac{k\pi x}{L}, \text{ and } \varphi_{2k}(x) = \sin\frac{k\pi x}{L} \text{ for } k = 1,2,3,\ldots \text{ and } 0 \le x \le L.\end{aligned}$$

Computation shows that

$$< \varphi_0, \varphi_0 >= \int_{-L}^{L} (\varphi_0)^2 = 2L, \text{ and}$$

$$< \varphi_k, \varphi_k >= \int_{-L}^{L} (\varphi_k)^2 = L \text{ for } k = 1,2,3,\ldots.$$

Example 6. Consider the problem

$$\begin{aligned} -\varphi'' &= \lambda\varphi \text{ on } [0,1], \\ \varphi(0) - \varphi'(0) &= 0, \text{ and} \\ \varphi(1) &= 0. \end{aligned}$$

This is a Sturm-Liouville problem, so it is self-adjoint and all eigenspaces are one-dimensional. If λ is an eigenvalue and φ is a corresponding real valued eigenfunction then (Rayleigh Quotient)

$$\begin{aligned} \lambda &= \frac{[(\varphi(0))^2 - \varphi'(1)\cdot 0] + \int_0^1 (\varphi')^2}{\int_0^1 (\varphi)^2}, \text{ so} \\ \lambda &= \frac{(\varphi(0))^2 + \int_0^1 (\varphi')^2}{\int_0^1 (\varphi)^2}. \end{aligned}$$

Thus all eigenvalues are nonnegative. The nonzero constant functions do not satisfy the boundary conditions, so all eigenvalues are positive.

The boundary conditions are equivalent to

$$M\begin{bmatrix} \varphi(0) \\ \varphi'(0) \end{bmatrix} + N\begin{bmatrix} \varphi(1) \\ \varphi'(1) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

where

$$M = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \text{ and } N = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

When $\lambda > 0$,

$$D(\lambda) = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & \sqrt{\lambda} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} \cos\sqrt{\lambda} & \sin\sqrt{\lambda} \\ -\sqrt{\lambda}\sin\sqrt{\lambda} & \sqrt{\lambda}\cos\sqrt{\lambda} \end{bmatrix}.$$

So

$$D(\lambda) = \begin{bmatrix} 1 & -\sqrt{\lambda} \\ \cos\sqrt{\lambda} & \sin\sqrt{\lambda} \end{bmatrix}$$

and

$$\Delta(\lambda) = \det D(\lambda) = \sin\sqrt{\lambda} + \sqrt{\lambda}\cos\sqrt{\lambda}.$$

From this we see that λ is an eigenvalue only in case

$$\lambda = \rho^2$$

where ρ is a positive number such that

$$\sin\rho + \rho\cos\rho = 0$$

or such that

$$\tan\rho = -\rho$$

(For each positive integer k there is exactly one solution to this equation between $k\pi - \dfrac{\pi}{2}$ and $k\pi + \dfrac{\pi}{2}$. See page 201 of the text. Newton's method (Look it up.) can be used to approximate the zeros of f where $f(\rho) = \sin\rho + \rho\cos\rho$.)

Note that when λ is an eigenvalue, then

$$D(\lambda) = \begin{bmatrix} 1 & -\sqrt{\lambda} \\ \cos\sqrt{\lambda} & \sin\sqrt{\lambda} \end{bmatrix} = \begin{bmatrix} 1 & -\sqrt{\lambda} \\ \cos\sqrt{\lambda} & -\sqrt{\lambda}\cos\sqrt{\lambda} \end{bmatrix}$$

so

$$D(\lambda)\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

only in case

$$c_1 - \sqrt{\lambda}c_2 = 0.$$

From this it follows that φ is a corresponding eigenfunction only in case

$$\varphi(x) = c\cos\sqrt{\lambda}x + \frac{c}{\sqrt{\lambda}}\sin\sqrt{\lambda}x$$

for all x in $[0,1]$ and some number $c \neq 0$.

Based on these observations, it follows that a proper listing of eigenvalues and eigenfunctions for this problem is $\{\lambda_k\}_{k=1}^{\infty}$ and $\{\varphi_k\}_{k=1}^{\infty}$ where $\lambda_k = \rho_k^2$, ρ_k is the kth positive number such that

$$\sin\rho + \rho\cos\rho = 0,$$

and

$$\varphi_k(x) = \cos\sqrt{\lambda_k}x + \frac{1}{\sqrt{\lambda_k}}\sin\sqrt{\lambda_k}x$$

for all x in $[0,1]$. Numerical approximations for the first three eigenvalues are as follows

k	ρ_k	λ_k
1	2.0288	4.1159
2	4.9132	24.139
3	7.9787	63.659

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