

MATH 3363 MIDTERM EXAM II. Sanders Fall '02

This exam has 5 problems and all 5 problems will be graded. You have the full hour and a half to complete it. Use my supplied paper only and return your solution sheets with the problems in order. Put your name, **last name first**, and **social security number** on each solution sheet you turn in. Good luck.

1. Solve **Laplace's equation** $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ on the unit square $0 < x < 1$, $0 < y < 1$ subject to the following boundary conditions.

$$(a) \quad \begin{aligned} u(x, 0) &= 0 & u(x, 1) &= 0 \\ u(0, y) &= 0 & u(1, y) &= \sin(\pi y) \end{aligned} \quad (c) \quad \begin{aligned} u(x, 0) &= 0 & u(x, 1) &= \sin(5\pi x) \\ u(0, y) &= 0 & u(1, y) &= \sin(\pi y) \end{aligned}$$

$$(b) \quad \begin{aligned} u(x, 0) &= 0 & u(x, 1) &= \sin(5\pi x) \\ u(0, y) &= 0 & u(1, y) &= 0 \end{aligned}$$

2. Solve the **heat equation** $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$ on the unit interval $0 < x < 1$ subject to the following boundary and initial conditions.

$$(a) \quad \begin{aligned} u(0, t) &= 0 & u_x(1, t) &= 0 \\ u(x, 0) &= \sin\left(\frac{5}{2}\pi x\right) \end{aligned} \quad (b) \quad \begin{aligned} u(0, t) &= 1 & u_x(1, t) &= 1 \\ u(x, 0) &= 1 + x + \sin\left(\frac{5}{2}\pi x\right) \end{aligned}$$

3. Solve the **wave equation** $\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}$ on the unit interval $0 < x < 1$ subject to following boundary and initial conditions.

$$(a) \quad \begin{aligned} u_x(0, t) &= 0 & u_x(1, t) &= 0 \\ u(x, 0) &= \cos(3\pi x) \\ u_t(x, 0) &= 0 \end{aligned} \quad (b) \quad \begin{aligned} u_x(0, t) &= 0 & u_x(1, t) &= 0 \\ u(x, 0) &= 0 \\ u_t(x, 0) &= 1 \end{aligned}$$

4. Solve the **2d heat equation** $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$ on the unit square $0 < x < 1$, $0 < y < 1$ subject to the following boundary and initial conditions.

$$(a) \quad \begin{aligned} u(x, 0, t) &= 0 & u(x, 1, t) &= 0 \\ u(0, y, t) &= 0 & u(1, y, t) &= 0 \\ u(x, y, 0) &= \sin(2\pi x) \sin(5\pi y) \end{aligned} \quad (b) \quad \begin{aligned} u(x, 0, t) &= 0 & u(x, 1, t) &= 0 \\ u(0, y, t) &= 0 & u(1, y, t) &= \sin(\pi y) \\ u(x, y, 0) &= \sinh(\pi x) \sin(\pi y) / \sinh(\pi) \end{aligned}$$

5. (This is a hard one. Skip it if you don't have a clue.)

Solve the **coupled system** of equations (find both $u(x, t)$ and $v(x, t)$).

$$\begin{aligned} \frac{\partial u}{\partial t} + \frac{\partial^2 v}{\partial x^2} &= 0 & \text{with boundary conditions:} & \quad \begin{aligned} u(0, t) &= u(1, t) = 0 \\ v(0, t) &= v(1, t) = 0 \end{aligned} \\ \frac{\partial v}{\partial t} - \frac{\partial^2 u}{\partial x^2} &= 0 & \text{and initial conditions:} & \quad \begin{aligned} u(x, 0) &= \sin(2\pi x) \\ v(x, 0) &= \sin(3\pi x) \end{aligned} \end{aligned}$$

Hint: You may use the fact that the general solution to the coupled system of ode's

$$\begin{aligned} \alpha_t - \gamma\beta &= 0 \\ \beta_t + \gamma\alpha &= 0 \end{aligned} \quad \text{is:} \quad \begin{aligned} \alpha(t) &= A \cos(\gamma t) + B \sin(\gamma t) \\ \beta(t) &= B \cos(\gamma t) - A \sin(\gamma t). \end{aligned}$$