MATH 3363 MIDTERM EXAM 2. Sanders Fall 2004

This exam has 5 problems and all 5 problems will be graded. You have one hour to complete it. Use my supplied paper only and return your solution sheets with the problems in order. Put your name, **last name first**, and **social security number** on each solution sheet you turn in. Good luck.

1. Consider the heat equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad \begin{array}{l}\text{with boundary conditions: } u(0,t) = 1,\ u(1,t) = 0, \\ \text{and initial condition: } u(x,0) = -x.\end{array}$$

(a) Determine the steady-state solution. (5 pts)

(b) Solve for $u(x,t)$. (15 pts)

2. Consider the wave equation

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}, \quad \begin{array}{l}\text{with boundary conditions: } u(0,t) = 1,\ u(1,t) = 1, \\ \text{and initial conditions: } u(x,0) = 1,\ u_t(x,0) = \sin(\pi x)\end{array}$$

(a) Determine the steady-state solution. (5 pts)

(b) Solve for $u(x,t)$ (15 pts).

3. Solve Laplace's equation $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$ on the unit square $0 \le x \le 1,\ 0 \le y \le 1$ with boundary conditions $u(0,y) = 0,\ u(1,y) = 0,\ u(x,0) = 0,\ u_y(x,1) = \sin(2\pi x)$.

4. Find the solution $u = u(r,\theta)$ to Laplace's equation $\dfrac{\partial^2 u}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial u}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2 u}{\partial \theta^2} = 0$ on the given domains satisfying the boundary conditions.

(a) The unit disk $r \le 1$ with $u(1,\theta) = 1 + \sin(2\theta)$.

(b) The annulus $1 \le r \le 2$ with $u(1,\theta) = 1$, and $u(2,\theta) = 2$.

5. Consider $\mathcal{L}(u) = \dfrac{d}{dx}\left(e^x \dfrac{du}{dx}\right) + u$ (written in Sturm-Liouville form) with boundary conditions $u(0) = 0$ and $u(1) = 0$.

(a) Prove all eigenvalues to $\mathcal{L}$ are real. (10 pts)

(b) Prove any two eigenfunctions to $\mathcal{L}$ associated to different eigenvalues are orthogonal with respect to $(u,v) = \int_0^1 u(x)v(x)dx$. (10 pts)