MATH 3363 MIDTERM EXAM III. Sanders Fall 2006

This exam has 5 problems, and all 5 problems will be graded. Use my supplied paper only. Return your solution sheets with the problems in order. Put your name, **last name first**, and **student id number** on each solution sheet you turn in. Each problem is worth 20 points with parts equally weighted unless otherwise indicated.

1. List all eigenvalues and eigenfunctions for the Laplacian $\nabla^2 u = u_{xx} + u_{yy}$ on the unit square $0 \le x, y \le 1$ satisfying the following boundary conditions.

(a) $\begin{array}{l} u(x,0,t) = u(x,1,t) = 0 \\ u(0,y,t) = u(1,y,t) = 0 \end{array}$ (b) $\begin{array}{l} u_y(x,0,t) = u_y(x,1,t) = 0 \\ u_x(0,y,t) = u_x(1,y,t) = 0 \end{array}$

2. Solve the 2 space dimensional **heat equation** $u_t = u_{xx} + u_{yy}$ on the unit square with

Boundary conditions: $u_y(x,0,t) = u_y(x,1,t) = 0 \quad u_x(0,y,t) = u_x(1,y,t) = 0$

Initial condition: $u(x,y,0) = 1 + \cos(3\pi x)\cos(4\pi y)$.

(The solution here has only two terms.)

3. Solve the 2 space dimensional **wave equation** $u_{tt} = u_{xx} + u_{yy}$ on the unit square with

Boundary conditions: $u(x,0,t) = u(x,1,t) = 0 \quad u(0,y,t) = u(1,y,t) = 0$

Initial conditions: $u(x,y,0) = 0 \quad u_t(x,y,0) = 1$.

(The solution here is an infinite series.)

4. Use the Fourier transform technique to solve the following first order equations on the interval $-\infty < x < \infty$. (Your answer should not involve the Fourier transform of f.)

(a) $\begin{array}{l} \dfrac{\partial u}{\partial t} + 2\dfrac{\partial u}{\partial x} = 0 \\ u(x,0) = f(x). \end{array}$ (b) $\begin{array}{l} \dfrac{\partial u}{\partial t} + \dfrac{\partial u}{\partial x} + u = 0 \\ u(x,0) = f(x). \end{array}$

5. Suppose $u(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{u}(\omega)e^{-i\omega x}d\omega$. Derive a formula for each $h(x)$ given below which **does not** involve the Fourier transform $\hat{u}(\omega)$.

(a) $h(x) = \dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{\infty} \hat{u}(\omega)e^{-i\omega(2x+1)}d\omega$ (c) $h(x) = \dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{\infty} (1-\omega^2)\,\hat{u}(\omega)e^{-i\omega x}d\omega$

(b) $h(x) = \dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{\infty} i\omega\,\hat{u}(\omega)e^{-i\omega(2x+1)}d\omega$ (d) $h(x) = \dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{\infty} \frac{1}{i\omega}\,\hat{u}(\omega)\left(e^{-i\omega x} - 1\right)d\omega$

(Hint for (d): Calculate $h'(x)$, observe that $h(0) = 0$ and use $h(x) - h(0) = \int_0^x h'(y)\,dy$.)