Math 3363 Sanders Spring 2008

Some Homework Questions for Exam 2

1. Show that the following operators have only real eigenvalues given that all are subject to the boundary conditions $u(0) = 0,\ u(1) = 0$.

(a) $\mathcal{L}(u) = (x^2+1)\dfrac{d^2u}{dx^2}$ (c) $\mathcal{L}(u) = \dfrac{d^2u}{dx^2} + \dfrac{du}{dx}$

(b) $\mathcal{L}(u) = \dfrac{d^2u}{dx^2} + (x^2+1)u$ (d) $\mathcal{L}(u) = (x^2+1)\dfrac{d^2u}{dx^2} + \dfrac{du}{dx}$

2. For each operator $\mathcal{L}$ with given boundary conditions from the previous exercise, determine a positive weight function $\omega(x)$ such that

$$\mathcal{L}(u) = \lambda u,\ \mathcal{L}(v) = \mu v,\ \text{with } \lambda \neq \mu\ \Rightarrow \int_0^1 u(x)v(x)\,\omega(x)dx = 0.$$

3(a). Calculate all eigenvalues and eigenfunctions for the operator $\mathcal{L}(u) = \dfrac{d^2u}{dx^2} + \dfrac{du}{dx}$ subject to boundary conditions $u(0) = 0,\ u(1) = 0$. (b) Expand the function $f(x) = 1$ into a Fourier series composed of these eigenfunctions. (You should need to calculate integrals $\int_0^1 e^{\frac{1}{2}x}\sin(n\pi x)\,dx$.)

4. Use the previous exercise to solve the following initial-boundary value problem on $(x,t) \in [0,1] \times [0,\infty)$.

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x} \text{ with } u(0,t) = 0,\ u(1,t) = 0, \text{ and } u(x,0) = 1.$$

5. Let Ω be a regular domain in $\mathbb{R}^2$ and consider the operator $\mathcal{L}(u) = \nabla^2 u$ on Ω where u satisfies the boundary condition $u|_{\partial\Omega} = 0$. (a) If u is an eigenvector with eigenvalue λ, show that λ must be a real number. (b) Moreover, show that all eigenvalues are strictly negative. (c) If $\nabla^2 u = \lambda u$ and $\nabla^2 v = \mu v$, $u|_{\partial\Omega} = 0$, $v|_{\partial\Omega} = 0$, and $\lambda \neq \mu$, show that $\int\int_\Omega uv\,dxdy = 0$.

6. Let Ω be a regular domain in $\mathbb{R}^2$. (a) Show that the boundary value problem $\nabla^2 u = f$, with f a given function defined on Ω, satisfying boundary condition $u|_{\partial\Omega} = g$, with g a given function defined on the boundary $\partial\Omega$, has at most one solution. (b) Show the same thing when the boundary condition is changed to $(u + \dfrac{\partial u}{\partial n})|_{\partial\Omega} = g$.

7. Solve Laplace's equation $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$ on the unit square $0 < x < 1,\ 0 < y < 1$ subject to the following boundary conditions.

(a) $u(x,0) = 0 \quad u(x,1) = 0$; $u(0,y) = 0 \quad u(1,y) = \sin(5\pi y)$

(b) $u(x,0) = 0 \quad u(x,1) = \sin(\pi x)$; $u(0,y) = 0 \quad u(1,y) = \sin(5\pi y)$

8. Solve Laplace's equation $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$ on the unit square $0 < x < 1,\ 0 < y < 1$ subject to the following boundary conditions.

(a) $u(x,0) = 0 \quad u_y(x,1) = \sin(2\pi x)$; $u(0,y) = 0 \quad u(1,y) = 0$

(b) $u(x,0) = 0 \quad u_y(x,1) = \sin(2\pi x)$; $u(0,y) = \sin(\frac{3}{2}\pi y) \quad u(1,y) = 0$

9. Consider the inhomogeneous initial-boundary value problem

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} - 2, \quad \begin{array}{l}\text{with boundary conditions: } u(0,t) = 1,\ u(1,t) = 0,\\ \text{and initial condition: } u(x,0) = x^2 - x + 1.\end{array}$$

(a) Determine the steady-state solution.

(b) Solve for $u(x,t)$.

10. Consider the inhomogeneous initial-boundary value problem

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = 0, \quad \begin{array}{l}\text{with boundary conditions: } u(0,t) = 1,\ u(1,t) = 2,\\ \text{and initial condition: } u(x,0) = x + 1,\ u_t(x,0) = \sin(\pi x)\end{array}$$

(a) Determine the steady-state solution.

(b) Solve for $u(x,t)$.