UH SUMMER SCHOOL 2016
HYPERBOLIC DYNAMICS AND BEYOND

DAY 1 PROBLEMS

Throughout these problems, $f : \mathbb{R} \to \mathbb{R}$ is a continuously differentiable homeomorphism and $E_\lambda : \mathbb{R} \to \mathbb{R}$ is defined by $E_\lambda(x) = \lambda x$.

Problem 1. This problem deals with the first linearization construction. Assume that $f(0) = 0$, $f(x) < x$ for every $x > 0$ and $f(x) > x$ for every $x > 0$.

(i) Show that $\lim_{n\to\infty} f^n(x) = 0$ for every $x \in \mathbb{R}$

(ii) The first construction of the lecture identifies a fundamental interval $I \subset \mathbb{R}$ for f and a corresponding interval $J \subset \mathbb{R}$ for E_λ. Then let h_0 be any orientation-preserving taking I to J, and h is the extension to $\mathbb{R}$ defined by:

$$h(x) = \lambda^n h_0(f^{-n}(x)),\ n \text{ is chosen so that } f^{-n}(x) \in I$$

Is h well-defined and continuous?

(iii) Can h_0 be chosen so that h is differentiable everywhere except possibly 0?

(iv) Can h_0 be chosen so that h is differentiable at 0? What if $\lambda = f'(0)$?

Problem 2. Show that if h is C^1 and $h \circ E_\lambda = E_\lambda \circ h$, then $h(x) = cx$ for some c. Find some h such that $h \circ E_{\lambda_1} = E_{\lambda_2} \circ h$. What are the differentiability properties of h if $\lambda_1 \neq \lambda_2$?

Problem 3. This problem deals with the second linearization construction. Recall that here, f is assumed to be twice differentiable, $f(0) = 0$ and $f'(0) = \lambda > 0$. Define $\varphi(x) = f(x)/(\lambda x)$ if $x \neq 0$.

(i) Show that φ extends continuously to 0

(ii) Show that the continuous extension of φ is continuously differentiable, and at 0, the derivative is $\varphi'(0) = f''(0)/(2\lambda)$

(iii) Show that the continuous solution obtained in the lecture is differentiable by differentiating directly.

Problem 4 (*). Find a continuously differentiable function $f : \mathbb{R} \to \mathbb{R}$ such that the h of the second linearization construction is continuous but not differentiable.

Problem 5. This problem deals with the third linearization construction. Assume that $f(x) = \lambda x + u(x)$, with $u(x)$ some function bounded above and below.

(i) Without using the linearization theorem, show that f has at least one fixed point

(ii) Show that if $|u'(x)| < |\lambda - 1|$, f has a unique fixed point