UH SUMMER SCHOOL 2016
HYPERBOLIC DYNAMICS AND BEYOND

DAY 2 PROBLEMS

Problem 1. Let $f, g : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ be maps of the circle. We take for granted the *lifting lemma*. That is, that there exist continuous functions $\tilde{f}, \tilde{g} : \mathbb{R} \to \mathbb{R}$ such that $f([x]) = [\tilde{f}(x)]$ (here, $[x]$ represents the equivalence class of $x \in \mathbb{R}$ modulo the integers).[1]

(i) Show that if $\tilde{f}_1$ and $\tilde{f}_2$ are two lifts of f, then $\tilde{f}_1 - \tilde{f}_2 \equiv m$ for some $m \in \mathbb{Z}$
(ii) Show that $\tilde{f}(x+1) - \tilde{f}(x) \equiv d$ for some d. d is called the *degree* of f.
(iii) Show that $\deg(f \circ g) = \deg(f) \cdot \deg(g)$
(iv) Show that if f is a homeomorphism, then $|\deg(f)| = 1$
(v) Show that if f is a continuous map of degree 1 and $\tilde{f}$ is a homeomorphism f is a homeomorphism
(vi) Give an example of a continuous map of degree 1 which is not a homeomorphism (just draw the graph)
(vii) Show that $\deg(f + g) = \deg(f) + \deg(g)$
(viii) Show that if $\deg(f) = 0$, then $\tilde{f}$ is 1-periodic
(ix) Using the previous 2 parts, show that any lift of f can be written as $\tilde{f}(x) = E_d(x) + \varphi(x)$ for some 1-periodic function $\varphi(x)$
(x) Show that if f is expanding, $|\deg(f)| \geq 2$

Problem 2. Using the formula for the conjugacy h between an expanding map f and its linear model E_d (coming from the Hartman-Grobman theorem), show that h is Hölder continuous. That is, show that:

$$d(h(x), h(y)) \leq C d(x, y)^{\alpha}$$

For some $\alpha > 0$. Estimate α.

Problem 3. Show that if $A : \mathbb{R}^d \to \mathbb{R}^d$ is a linear map, if all eigenvalues of A have modulus > 1, then there exists a norm on $\mathbb{R}^d$ such that $\left|\left|A^{-1}\right|\right| < 1$. Furthermore, show that if there exists a norm for which $\left|\left|A^{-1}\right|\right| < 1$, then for any other norm, $||A^{-n}||' < 1$ for some n (which depends on the norm).

Problem 4. A *Riemannian metric* on $\mathbb{R}^d$ is a function $g : \mathbb{R}^d \to \mathcal{PD}_d$, where $\mathcal{PD}_d$ is the space of $d \times d$ symmetric positive definite matrices. Then given a vector $v \in \mathbb{R}^d$, we can define its norm at x to be $||v||_x = v^T g(x) v$ and the inner product of v and w as $\langle v, w \rangle_x = v^T g(x) w$. Show that if f^n is expanding with respect to some Riemannian metric g, then f is expanding with respect to the *adapted metric* (for some $\lambda < 1$)[2]:

$$g_\lambda(x) = \sum_{n \geq 0} \lambda^n (Df^n_x)^{-1} (g \circ f^n(x)) (Df^n_x)$$

Problem 5. Show that the conjugating homeomorphism of the Hartman-Grobman theorem is a homeomorphism in $\mathbb{R}^d$ with $d \geq 2$

[1]While intuitively clear, this is a nontrivial topological lemma which requires some additional background
[2]First, show that this converges for nice λ and is a metric

Problem 6 (*)**.** This problem outlines an alternate, geometric proof that all expanding maps are topologically conjugate to E_d for some d. Let $f : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ be an expanding map of degree d.

(i) Show that the sets $D_m = [m/d, (m+1)/d]$ have the property that $E_d(D_m) = \mathbb{R}/\mathbb{Z}$ and $E_d(\mathrm{Int}(D_m)) = \mathbb{R}/\mathbb{Z} \setminus \{0\}$

(ii) Show that the sets $D_{m,n} = [m/d^n, (m+1)/d^n]$ have the property that $E_d(D_{m,n}) = D_{m,n-1}$ if $n \geq 2$

(iii) Show that for each $x \in \mathbb{R}/\mathbb{Z}$, $x = \bigcap_{n=1}^{\infty} D_{m_n,n}$ for some sequence m_n with $D_{m_{n+1},n+1} \subset D_{m_n,n}$. Furthermore, show that m_n is either unique, or x takes the form c/d^N for some $c, N \in \mathbb{N}$ and the sequence m_n can be chosen in at most 2 ways.

(iv) Show that the sequence m_n satisfies $x = \lim_{n\to\infty} \frac{m_n}{d^n}$

(v) Show that there are intervals $\Delta_m \subset \mathbb{R}/\mathbb{Z}$ such that:
  (a) The left endpoint on Δ_0 is a fixed point for f
  (b) $f(\Delta_m) = \mathbb{R}/\mathbb{Z}$
  (c) $f(\mathrm{Int}(\Delta_m)) = \mathbb{R}/\mathbb{Z} \setminus \{0\}$

(vi) Show that there exist intervals $\Delta_{m,n}$ such that:
  (a) the right endpoint of $\Delta_{m,n}$ is the left endpoint of $\Delta_{m+1,n}$
  (b) $\Delta_{m,1} = \Delta_m$
  (c) If $n \geq 2$, $f(\Delta_{m,n}) = \Delta_{m+1,n}$

(vii) Show that the length of $\Delta_{m,n}$ is less than $c\lambda^n$ for some $\lambda < 1$ [*Hint*: Use the expansivity of f]

(viii) Show that if $\bigcap_{n\geq 1} \Delta_{m_n,n} = \{x_0\}$, then $\bigcap_{n\geq 1} D_{m_n,n} = \{x_1\}$ is also a singleton.

(ix) Show that if we define $h(x_0) = x_1$, then h is well-defined, continuous and conjugates the dynamics of f and h [*Hint*: For continuity, use part (vii)]

(x) (**) Use these to show that any expanding map is topologically semiconjugate to $\sigma : \Sigma_d^+ \to \Sigma_d^+$, the full (one-sided) shift on d symbols (the semiconjugacy will be a map $h : \Sigma_d \to S^1$ which intertwines the dynamics).