UH SUMMER SCHOOL 2016
HYPERBOLIC DYNAMICS AND BEYOND

DAY 3 PROBLEMS: THE LAST DAY ☹

Problem 1. If X is a topological space, a *vector bundle* over X is a function $V : X \to \mathrm{Gr}(d,k)$[1]. The *fiber* over $x \in X$ is the space $V_x = V(x)$. A *section* of a vector bundle is a map $v : X \to \mathbb{R}^d$ such that $v(x) \in V_x$ for every $x \in X$.

(i) Show that if at each point $x \in X$, $\mathbb{R}^d = E^s_x \oplus E^u_x$, then any $\mathbb{R}^d$-valued function $\delta : X \to \mathbb{R}^d$ can be written as $\delta = \delta^s + \delta^u$, where δ^σ is section of the vector bundle E^σ

(ii) Show that if V is continuous, then $\mathcal{B} = \{(x,v) : v \in V(x)\}$ is a closed subspace of $X \times \mathbb{R}^d$ ($\mathcal{B}$ is called the *total space*) and the projection $\pi : \mathcal{B} \to X$ defined by $\pi(x,v) = x$ has $\pi^{-1}(x) = V_x$

(iii) A bundle V is called *invariant* under a dynamical system $f : X \to X$ and a cocycle A_x if $A_x V(x) \subset V(f(x))$. If $T : \mathcal{B} \to \mathcal{B}$ is defined by $T(x,v) = (f(x), A_x v)$, show that T is well-defined, continuous, and that $T^n(x,v) = (f^n(x), A^{(n)}_x(v))$

(iv) Show that if $V = V^1 \oplus V^2$ is a sum of two subbundles, V^1 and V^2 are invariant under f and A, and $\pi^i_x : V_x \to V^i_x$ is the projection of V onto V_i, then $A_x \circ \pi^i_x = \pi^i_{f(x)} \circ A_x$

(v) Assume that $\mathbb{R}^d = E^s \oplus E^u$, A_x is invertible and $||A_x|_{E^s}|| < \lambda < 1$ and $\left|\left|A_x^{-1}|_{E^u}\right|\right| < \lambda < 1$. Let B_x be a cocycle such that $||A_x - B_x|| < \varepsilon$. If V is a vector bundle, let $C(V,\delta) = \{v : \angle(v,V) < \delta\}$.
   (a) Show that for sufficiently small ε, δ $B_x(\overline{C(E^s_x,\delta)}) \subset C(E^s_{f(x)},\delta)$
   (b) Show that if $d_{C^0}(f,g)$ is sufficiently small, then the same property holds
   (c) Show that $\bigcap_{n\le 0} B_x^{-n}(\overline{C(E^s_{g^n(x)},\delta)})$ is a subspace of dimension $\dim E^s$

Problem 2. Let Σ_d be the space of (2-sided) sequences on the alphabet $\{1,\dots,d\}$ equipped with the metric:

$$d((x_n),(y_n)) = \sum_{n=-\infty}^{\infty} 2^{-|n|}\delta(x_n,y_n)$$

where $\delta(a,b) = 1$ if $a = b$ and 0 otherwise. Show that the shift $\sigma : \Sigma_d \to \Sigma_d$ satisfies the same shadowing property and Anosov closing lemma proved in lecture for hyperbolic sets.

[1]Technically, this definition only gives subbundles of a trivial bundle, but often this is enough

Problem 3 (Invertible Extensions)**.** Given $f : Y \to Y$ a continuous dynamical system on a compact metric space Y, we construct a compact metric space X_f as:

$$X_f = \{(\dots, x_{-1}, x_0, x_1, \dots) : x_{i+1} = f(x_i)\} \subset Y^{\mathbb{Z}}$$

(i) Show that X_f is a closed, σ-invariant (and hence compact) subset of $Y^{\mathbb{Z}}$
(ii) Show that f is a factor $\sigma : X_f \to X_f$ (ie, there is a semiconjugacy $h : X_f \to Y$)
(iii) Show that f and σ are topologically conjugate if f is invertible
(iv) (*) Let $f = E_2 : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$. Show that σ is topologically conjugate to the Smale-Williams solenoid. An exact formula for a Smale-Williams Solenoid system $f : D^2 \times S^1 \to D^2 \times S^1$ is:

$$f(x, y) = \left(\frac{1}{10} x + \frac{1}{2} i(y), 2y \right)$$

Where $i : S^1 \to D^2$ is the inclusion of S^1 to the boundary. [*Hint*: Construct a homeomorphism h_n from the space of sequences starting from index $-n$ to $f^n(\{0\} \times S^1)$ such that in the second coordinate, $h_{n-1} \circ \sigma = f \circ h_{n-1}$. Then take a limit in $C^0(X_f, D^2 \times S^1)$]