MR1750438 (2001j:37062) 37D25 37A25 37D50
Young, Lai-Sang (1-UCLA)
Recurrence times and rates of mixing. **(English summary)**
Israel J. Math. **110** (1999), 153–188.

This paper concerns the statistical properties of nonuniformly hyperbolic dynamical systems. The author already introduced in [Ann. of Math. (2) **147** (1998), no. 3, 585–650; MR1637655 (99h:58140)] a powerful scheme to study properties such as the existence of Sinaĭ-Bowen-Ruelle measure, the decay of correlations and the central limit theorem for systems with some hyperbolic behavior.

The strategy proposed by Young is that if a system has some hyperbolicity, then one has a chance to construct a Markov tower having the original system as a factor, hence all the above-mentioned statistical properties may be recovered from those in the tower. The Markov tower is an abstract system defined as follows: Let $F\colon \Delta_0 \to \Delta_0$ be a map and R an integer-valued function defined on Δ_0. Assume that Δ_0 is partitioned into countably many elements $\Delta_{0,i}$ such that $R = R_i$ is constant on each $\Delta_{0,i}$ and F^{R_i} maps $\Delta_{0,i}$ onto Δ_0 bijectively. Then the Markov tower (Δ, F) is the discrete time special semi-flow over (Δ_0, F^R) with height function R. The problem is now to study these properties on the tower. In the author's earlier paper [op. cit.] this was done for exponential mixing rates, using spectral arguments. Here she continues the study of this abstract model, and considers the case of sub-exponential mixing rates. This is of great interest, since for nonuniformly hyperbolic systems it is unlikely that this rate is exponential.

The main result is that if m is a nonsingular probability measure, then, under some mild distortion conditions, the speed of convergence to equilibrium is of the same type as the speed of convergence to zero of the tail of the distribution of the return time $m(R > n)$. For example, if $m(R > n) = O(n^{-\alpha})$ then the decay of correlation occurs with a speed $O(n^{-\alpha})$. This upper bound on the speed of convergence to equilibrium is obtained by a probabilistic coupling method. Very quickly, let P be a probability measure on $\Delta \times \Delta$ which is the product of two measures which are equivalent to m. Then if $T(x, y)$ denotes the simultaneous return time by F of $(x, y) \in \Delta \times \Delta$ to $\Delta_0 \times \Delta_0$, the speed of convergence to equilibrium is essentially related to the tail $P(T > n)$. Note that this is a well-known upper bound for the speed of convergence to equilibrium of a countable state Markov chain [see, e.g., T. Lindvall, *Lectures on the coupling method*, Wiley, New York, 1992; MR1180522 (94c:60002)]. In fact, the real bound obtained in the theorem involves the distribution of some successive simultaneous return times, and can hardly be written down here. Fortunately Young computed it for us for polynomial, exponential, and stretched exponential tails, and found a rate of convergence of the same type.

As an alternative to this coupling approach, V. Maume-Deschamps [Trans. Amer. Math. Soc. **353** (2001), no. 8, 3371–3389 (electronic) MR1828610 (2002a:37004)] was able to get, using Hilbert projective metrics, a simple upper bound for the decay of correlations in terms of the measure of high levels of the tower.

In the last part of the paper, the author applies this method to one-dimensional maps with indifferent fixed points, and essentially recovers for a broader class of systems the results obtained earlier on by C. Liverani, Saussol and S. Vaienti, and H. Hu.

Markov towers seem to be particularly indicated to model nonuniformly hyperbolic systems, anologously to the way in which Markov partitions model Axiom-A systems. This opens a promising path to studying systematically the statistical properties of hyperbolic measures in the setting of Pesin's theory.

REVISED (December, 2001)
Current version of review. Go to earlier version. *Benoit Saussol*

References

1. M. Benedicks and L.-S. Young, *Decay or correlations for certain Henon maps*, to appear in Astérisque, in a volume in honor of Douady. MR1755436 (2001g:37030)
2. A. Fisher and A. Lopes, *Polynomial decay of correlation and the central limit theorem for the equilibrium state of a non-Hölder potential*, preprint, 1997. cf. MR 2002g:37032
3. F. Hofbauer and G. Keller, *Ergodic properties of invariant measures for piecewise monotonic transformations*, Mathematische Zeitschrift **180** (1982), 119–140. MR0656227 (83h:28028)
4. H. Hu, *Decay of correlations for piecewise smooth maps with indifferent fixed points*, preprint. cf. MR 2005a:37064
5. H. Hu and L.-S. Young, *Nonexistence of* SBR *measures for some systems that are "almost Anosov"*, Ergodic Theory and Dynamical Systems **15** (1995), 67–76. MR1314969 (95j:58096)
6. S. Isola, *On the rate of convergence to equilibrium for countable ergodic Markov chains*, preprint, 1997. cf. MR 2005c:60089
7. C. Kipnis and S.R.S. Varadhan, *Central limit theorem for additive functions of reversible Markov process and applications to simple exclusions*, Communications in Mathematical Physics **104** (1986), 1–19. MR0834478 (87i:60038)
8. C. Liverani, *Decay of correlations*, Annals of Mathematics **142** (1995), 239–301. MR1343323 (96e:58090)
9. C. Liverani, *Central limit theorem for deterministic systems*, International Conference on Dynamical Systems, Montevideo 1995 (F. Ledrappier, J. Lewowicz and S. Newhouse, eds.), Pitman Research Notes in Mathematics **362** (1996), 56–75. MR1460797 (98k:28025)
10. C. Liverani, B. Saussol and S. Vaienti, *A probabilistic approach to intermittency*, preprint. cf. MR 2000d:37029
11. R. Mañé, *Ergodic Theory and Differentiable Dynamics*, Springer-Verlag, Berlin, 1983. MR0889254 (88c:58040)
12. G. Pianigiani, *First return maps and invariant measures*, Israel Journal of Mathematics **35** (1980), 32–48. MR0576460 (82a:58031)
13. M. Pollicott, *Rates of mixing for potentials of summable variation*, to appear in Transactions of the American Mathematical Society. MR1621698 (2000c:37005)
14. J.W. Pitman, *Uniform rates of convergence for Markov chain transition probabilities*, Zeitschrift für Wahrscheinlichkeitstheorie und Verwandte Gebiete **29** (1974), 193–227. MR0373012 (51 #9214)
15. D. Ruelle, *Thermodynamic Formalism*, Addison-Wesley, New York, 1978. MR0511655 (80g:82017)
16. P. Tuominen and R. Tweedie, *Subgeometric rates of convergence of f-ergodic Markov chains*, Advances in Applied Probability **26** (1994), 775–798. MR1285459 (95m:60097)
17. L.-S. Young, *Statistical properties of dynamical systems with some hyperbolicity*, Annals of Mathematics **147** (1998), 558–650. MR1637655 (99h:58140)

Note: This list reflects references listed in the original paper as accurately as possible with no attempt to correct errors.

MR1637655 (99h:58140) 58F15 58F11
Young, Lai-Sang (1-UCLA)
Statistical properties of dynamical systems with some hyperbolicity.
Ann. of Math. (2) **147** (1998), *no. 3,* 585–650.

FEATURED REVIEW.

This paper contains a breakthrough result in the study of the decay of correlations for hyperbolic systems with singularities. Hyperbolic systems (characterized by the absence of zero Lyapunov exponents) include many classical and modern models of statistical physics: Lorentz gases, hard ball systems, certain chaotic attractors. While smooth uniformly hyperbolic systems are very well understood, this is not the case for many physical models, which are not uniformly hyperbolic and contain singularities in the phase space. One of the most important issues here is the behavior of the time correlation function, which is directly related to the basic laws of statistical mechanics.

It was long conjectured that singularities in hyperbolic systems, e.g., in Lorentz gases and other Sinaĭ billiards, might slow down the decay of correlations from exponential to subexponential. That would imply that statistical properties of these models are weak and physical laws could be hard to prove or even observe experimentally. Many researchers have computed the correlation function numerically, but the data obtained were compatible with both exponential and subexponential laws, and the controversy persisted for more than a decade. This paper proves that correlations actually decay exponentially for hyperbolic maps with singularities under quite general assumptions. The main result of this paper covers certain Sinaĭ billiards, including planar periodic Lorentz gases with finite horizon, and this is also shown in the paper.

The content of the main theorem is the following, technical details being left out. Let $f\colon M\to M$ be a nonuniformly hyperbolic map, possibly with singularities. Let $\Gamma\subset M$ be a "hyperbolic product structure", i.e. the intersection between a family of unstable manifolds and a family of stable manifolds. The set Γ is iterated under $f^n,\ n\geq 1$, until its images properly cross Γ itself (in the sense that the intersection stretches completely across Γ in the unstable direction). Every time a proper intersection occurs, the points of $f^n\Gamma$ coming back to Γ "make a return". The points that make a return are taken out of circulation, and the rest of $f^n\Gamma$ is iterated further, until proper intersections with Γ occur again, etc. The main assumption of the theorem is that the fraction of Γ that does not make a return until time n is exponentially small in n (with respect to the Lebesgue measure on unstable manifolds). Under this assumption, the author proves that an f-invariant Sinaĭ-Ruelle-Bowen measure on M exists, enjoys exponential decay of correlations and satisfies a central limit theorem.

The author applies the main theorem to dispersing Sinaĭ billiards, Axiom A attractors and logistic maps of the interval. The theorem has potentially wide applications to other chaotic deterministic systems of physical origin. *Nikolai Chernov*

References

1. M. Benedicks and L. Carleson, On iterations of $1-ax^2$ on $(-1,1)$, Ann. of Math. **122** (1985), 1–25. MR0799250 (87c:58058)
2. M. Benedicks, L. Carleson, The dynamics of the Hénon map, Ann. of Math. **133** (1991), 73–169. MR1087346 (92d:58116)
3. M. Benedicks and L.-S. Young, Absolutely continuous invariant measures and ran-

dom perturbations for certain one-dimensional maps, Ergod. Th. & Dynam. Sys. **12** (1992), 13–37. MR1162396 (93d:58087)

4. M. Benedicks, L.-S. Young, SBR measures for certain Hénon maps, Invent. Math. **112** (1993), 541–576. MR1218323 (94e:58074)
5. M. Benedicks, L.-S. Young, Decay or correlations for certain Hénon maps, 1996 preprint. cf. MR 97i:58100
6. R. Bowen, *Equilibrium States and the Ergodic Theory of Anosov Diffeomorphisms*, Springer Lecture Notes in Math. **470** (1975). MR0442989 (56 #1364)
7. L. A. Bunimovich, Ya. G. Sinai and N. I. Chernov, Markov partitions for two-dimensional hyperbolic billiards, Russ. Math. Surv. **45** (1990), 105–152. MR1071936 (91g:58155)
8. L. A. Bunimovich, Ya. G. Sinai, N. I. Chernov, Statistical properties of two-dimensional hyperbolic billiards, Russ. Math. Surv. **46** (1991), 47–106. MR1138952 (92k:58151)
9. N. I. Chernov, Ergodic and statistical properties of piecewise linear hyperbolic automorphisms of the 2-torus, J. of Stat. Phys. **69** (1992), 111–134. MR1184772 (94d:58085a)
10. N. I. Chernov, Limit theorems and Markov approximations for chaotic dynamical systems, Prob. Theory Related Fields **101** (1995), 321–362. MR1324089 (96m:28016)
11. N. I. Chernov, G. Eyink, J. Lebowitz and Ya. G. Sinai, Steady-state electrical conduction in the periodic Lorentz gas, Comm. Math. Phys. **154** (1993), 569–601. MR1224092 (94k:82058)
12. P. Collet and J.-P. Eckmann, Positive Liapunov exponents and absolute continuity for maps of the interval, Ergod. Th. & Dynam. Sys. **3** (1983), 13–46. MR0743027 (85j:58092)
13. P. Collet and Y. Levy, Ergodic properties of the Lozi mappings, Comm. Math. Phys. **93** (1984), 461–481. MR0763754 (86d:58067)
14. N. Dunford and J. T. Schwartz, *Linear Operators, Part* I: *General Theory*, 1957. MR1009162 (90g:47001a)
15. M. I. Gordin, The central limit theorem for stationary processes, Soviet Meth. Dokl. **10** (5) (1969), 1174–1176. MR0251785 (40 #5012)
16. F. Hofbauer and G. Keller, Ergodic properties of invariant measures for piecewise monotonic transformations, Math. Z. **180** (1982), 119–140. MR0656227 (83h:28028)
17. H. Hu, Conditions for the existence of SRB measures for "almost Anosov" diffeomorphisms, 1995 preprint.
18. H. Hu and L.-S. Young, Nonexistence of SBR measures for some diffeomorphisms that are "almost Anosov", Ergod. Th. & Dynam. Sys. **15** (1995), 67–76. MR1314969 (95j:58096)
19. M. Jakobson, Absolutely continuous invariant measures for one-parameter families of one-dimensional maps, Comm. Math. Phys. **81** (1981), 39–88. MR0630331 (83j:58070)
20. A. Katok and J.-M. Strelcyn, *Invariant Manifolds, Entropy and Billiards; Smooth Maps with Singularities*, Springer Lecture Notes in Math. **1222** (1986). MR0872698 (88k:58075)
21. C. Kipnis and S.R.S. Varadhan, Central limit theorem for additive functionals of reversible Markov processes and applications to simple exclusions, Comm. Math. Phys. **104** (1986), 1–19. MR0834478 (87i:60038)
22. G. Keller, Un théorème de la limite centrale pour une classe de transformations monotones par morceaux, Comptes Rendus de l'Académie des Sciences, Série A **291** (1980), 155–158. MR0605005 (82i:60042)
23. G. Keller and T. Nowicki, Special theory, zeta functions and the distribution of

periodic points for Collet-Eckmann maps, Comm. Math. Phys. **149** (1992), 31–69. MR1182410 (93i:58123)
24. A. Lasota and J. Yorke, On the existence of invariant measures for piecewise monotonic transformations, Trans. A. M. S. **186** (1973), 481–488. MR0335758 (49 #538)
25. F. Ledrappier, Propriétés ergodiques des mesures de Sinai, Publ. Math. IHES **59** (1984), 163–188. MR0743818 (86f:58092)
26. C. Liverani, Decay of correlations, Ann. of Math. **142** (1995), 239–301. MR1343323 (96e:58090)
27. C. Liverani, Central limit theorem for deterministic systems, Int. Conf. of Dyn. Sys., Montevideo, 1995, Eds. F. Ledrappier, J. Lewowicz, S. Newhouse, Pitman Research Notes in Math. **362** (1996), 56–75. MR1460797 (98k:28025)
28. L. Mora and M. Viana, Abundance of strange attractors, Acta Math. **171** (1993), 1–71. MR1237897 (94k:58089)
29. T. Nowicki and S. van Strein, Absolutely continuous invariant measures under a summability condition, Invent. Math. **105** (1991), 123–136. MR1109621 (93b:58094)
30. W. Parry and M. Pollicott, *Zeta Functions and the Period Orbit Structure of Hyperbolic Dynamics*, Soc. Math. de France (Astérisque **187-188**), Paris, 1990. MR1085356 (92f:58141)
31. Ya. B. Pesin, Characteristic Lyapunov exponents and smooth ergodic theory, Russ. Math. Surveys **32** (1977), 55–114. MR0466791 (57 #6667)
32. Ya. B. Pesin, Dynamical systems with generalized hyperbolic attractors: hyperbolic, ergodic and topological properties, Ergod. Th. & Dynam. Sys. **12** (1992), 123–151. MR1162404 (93b:58095)
33. C. Pugh and M. Shub, Ergodic attractors, Trans. A. M. S. **312** (1989), 1–54. MR0983869 (90h:58057)
34. D. Ruelle, A measure associated with Axiom-A attractors, Amer. J. Math. **98** (1976), 619–654. MR0415683 (54 #3763)
35. D. Ruelle, *Thermodynamic Formalism*, Addison-Wesley, New York, 1978. MR0511655 (80g:82017)
36. D. Ruelle, Flots qui ne mélangent pas exponentiellement, C. R. Acad. Sci. Paris Sér. I Math. **296** (1983), 191–193. MR0692974 (84h:58085)
37. M. Rychlik, Invariant measures and the variational principle for Lozi mappings, Ph.D. Thesis (unpublished), Berkeley, 1983. MR0691018 (84b:58065)
38. Ya. G. Sinai, Gibbs measures in ergodic theory, Russ. Math. Surveys **27** (1972), 21–69. MR0399421 (53 #3265)
39. Ya. G. Sinai, Dynamical systems with elastic reflections: Ergodic properties of dispersing billiards, Russ. Math. Surveys **25** (1970), 137–189. MR0274721 (43 #481)
40. P. Thieullen, C. Tresser and L.-S. Young, Positive Lyapunov exponent for generic one-parameter families of unimodal maps, C. R. Acad. Sci. Paris, t. 315 Série I (1992), 69–72; J'Analyse **64** (1994), 121–172. MR1303510 (95g:58134)
41. M. Wojtkowski, Invariant families of cones and Lyapunov exponents, Ergod. Th. & Dynam. Sys. **5** (1985), 145–161. MR0782793 (86h:58090)
42. L.-S. Young, A Bowen-Ruelle measure for certain piecewise hyperbolic maps, Trans. A. M. S. **287** (1985), 41–48. MR0766205 (86e:58052)
43. L.-S. Young, Decay of correlations for certain quadratic maps, Comm. Math. Phys. **146** (1992), 123–138. MR1163671 (93j:58083)
44. L.-S. Young, Recurrence times and rates of mixing, 1997 preprint. cf. MR 2001j:37062