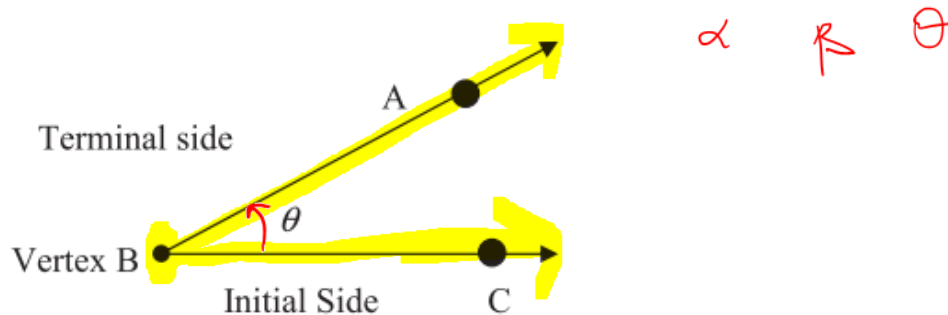


Math 1330 - Section 4.2 Radians, Arc Length, and Area of a Sector

The word trigonometry comes from two Greek roots, trigonon, meaning “having three sides,” and meter, meaning “measure.” We have already defined the six basic trigonometric functions in terms of a right triangle and the measures of its three sides.

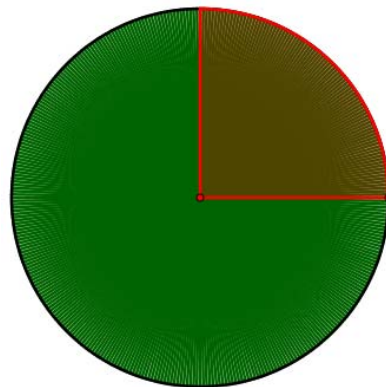
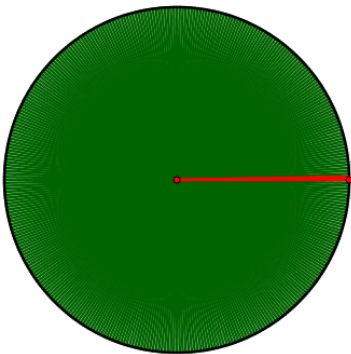
Before beginning our study of trigonometry, we need to take a look at some basic concepts having to do with angles.

An angle is formed by two rays that share a common endpoint, called the **vertex** of the angle. One ray is called initial side of the angle, and the other side is called the terminal side. For ease, we typically will draw angles in the coordinate plane with the initial side along the positive x axis.



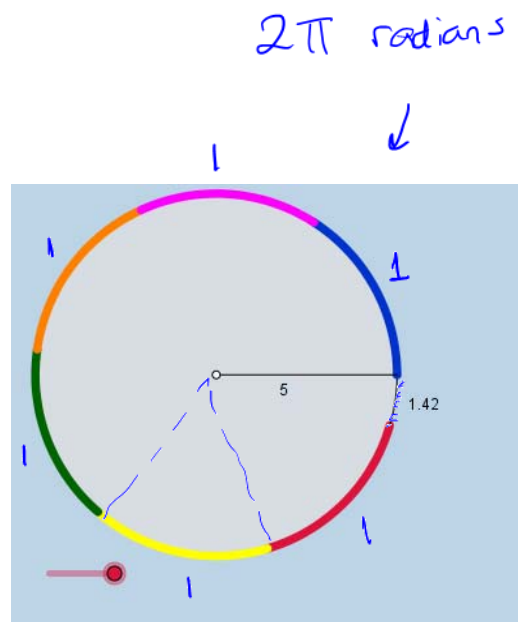
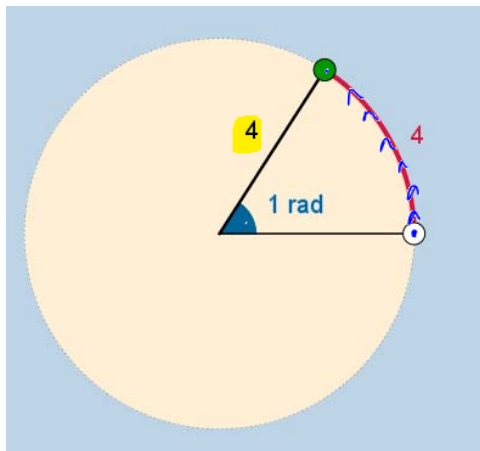
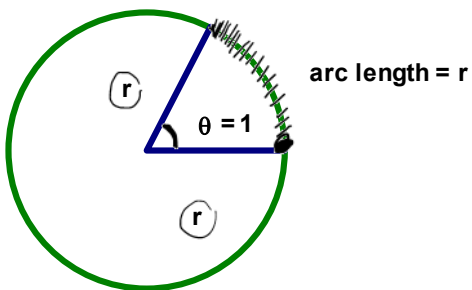
We measure angles in two different ways, both of which rely on the idea of a complete revolution in a circle.

You are probably familiar with degree measure. In this system of angle measure, an angle which is one complete revolution is 360° . So one degree is $\frac{1}{360}$ th of a circle.



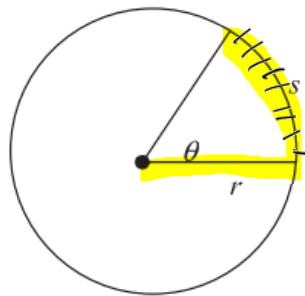
The second method is called **radian measure**. One complete revolution is 2π .

Suppose we draw a circle and construct an angle by drawing rays from the center of the circle to two different points on the circle in such a way that the length of the arc intercepted by the two rays is the same as the radius of the circle. The measure of the central angle thus formed is **one radian**.



Radian measure of an angle:

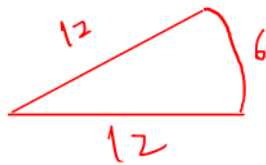
Place the vertex of the angle at the center of a circle of radius r . Let s denote the length of the arc intercepted by the angle. The **radian measure** θ of the angle is the ratio of the arc length s to the radius r . That is, $\theta = \frac{s}{r}$.



$$\theta = \frac{s}{r}$$

In general, the **radian measure of a central angle** θ can be determined by the formula $\theta = \frac{s}{r}$, where s is the length of the intercepted arc and r is the radius of the circle and r and s are measured in the same units.

Example 1: A circle has radius 12 inches. A central angle θ intercepts an arc of length 6 inches. What is the radian measure of θ ?



$$\theta = \frac{s}{r} = \frac{6}{12} = \frac{1}{2} \text{ radians.}$$

We know that the circumference of a circle is $2\pi r$. In this case, $\theta = \frac{2\pi r}{r} = 2\pi$. So the radian measure of the central angle in the case of a complete revolution is 2π .

Comparing the two systems, then, we have that

$$1^\circ = \frac{\pi}{180} \text{ rad.} \quad \leftarrow \begin{array}{l} 2\pi \text{ radians} = 360^\circ \\ \pi \text{ radians} = 180^\circ \\ \frac{\pi}{2} \text{ radians} = 90^\circ \end{array} \rightarrow 1 \text{ rad} = \frac{180^\circ}{\pi}$$

etc.

As you are becoming more familiar with radian measure, you may find it helpful to be able to convert between the two systems. We can use the statement π radians = 180°

to help do this. Dividing both sides of that equation by π , we have that

$$1 \text{ radian} = \frac{180^\circ}{\pi} \text{ so } \underline{\text{to convert to degrees, multiply by } \frac{180^\circ}{\pi}}.$$

Similarly,

$$1 \text{ degree} = \frac{\pi}{180^\circ} \text{ so } \underline{\text{to convert to radians, multiply by } \frac{\pi}{180^\circ}}.$$

These are the conversion formulas for radians to degrees and for degrees to radians, respectively.

$$1 \text{ radian} = \frac{180^\circ}{\pi} \text{ so to convert to degrees, multiply by } \frac{180^\circ}{\pi}.$$

$$1 \text{ degree} = \frac{\pi}{180^\circ} \text{ so to convert to radians, multiply by } \frac{\pi}{180^\circ}.$$

Example 2: Convert 135° to radian measure.

$$\overset{3}{\cancel{135}^\circ} \cdot \frac{\pi}{\cancel{180}^\circ_4} = \frac{3 \cdot \pi}{4} \text{ radians}$$

Example 3: Convert $\frac{4\pi}{3}$ to degrees.

$$\frac{\cancel{4\pi}}{\cancel{3}} \cdot \frac{\overset{60}{\cancel{180}^\circ}}{\cancel{\pi}} = 240^\circ$$

Example 4: Convert $\frac{2\pi}{9}$ to degrees.

$$\frac{\cancel{2\pi}}{\cancel{9}} \cdot \frac{\overset{20^\circ}{\cancel{180}^\circ}}{\cancel{\pi}} = 40^\circ$$

Example 5: Convert 18° to radian measure.

$$\cancel{18}^\circ \cdot \frac{\pi}{\cancel{180}^\circ_{10}} = \frac{\pi}{10} \text{ radians}$$

You will use some angles so often that you should know both their degree and radian measures. These are:

$30^\circ = \frac{\pi}{6}$
 $45^\circ = \frac{\pi}{4}$
 $60^\circ = \frac{\pi}{3}$
 $90^\circ = \frac{\pi}{2}$
 $180^\circ = \pi$
 $360^\circ = 2\pi$

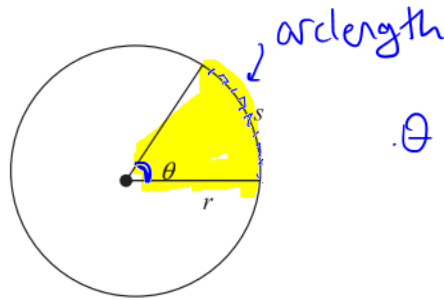
$\frac{2\pi}{3}$ $\frac{5\pi}{6}$...

Memorize these!

POPPER FOR SECTION 4.2:

Question#1: Convert $\frac{3\pi}{4}$ radians to degrees.

Choices on the video



$$\theta = \frac{s}{r} \Rightarrow s = r \cdot \theta$$

If $\theta = \frac{s}{r}$, then we can multiply both sides of this equation by r , which gives us $s = r\theta$.

This is called the **arclength formula** and it gives the length of the arc intercepted by the central angle. Note, to use this formula the angle measure **MUST** be given **in radians**. ★

Example 6: If the radius of a circle is 16 inches and the measure of its central angle is $\frac{\pi}{4}$, find the arclength of the sector intercepted by the angle.

$$s = r \cdot \theta = 16 \cdot \frac{\pi}{4} = 4\pi$$

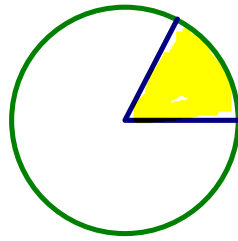


Example 7: If the arclength of a sector is 8π cm. and the radius is 12 cm., find the measure of the central angle.

$$s = 8\pi \quad r = 12$$

$$\theta = \frac{s}{r} = \frac{8\pi}{12} = \frac{2\pi}{3} \text{ radians}$$

A **sector of a circle** is the region bounded by a central angle and the intercepted arc.



$$A = \frac{1}{2} \cdot r^2 \cdot \theta$$

↑
in
radians

Sometimes, you'll need to find the **area of a sector**. The formula for the area of a circle is $A = \pi r^2$. A sector is a fraction of a circle, determined by the measure of its central angle over the complete revolution that is a circle, that is $\frac{\theta}{2\pi}$. So the area of a section is this fraction of the area of the circle, that is:

Must
learn

$$\underline{A = \pi r^2 \cdot \frac{\theta}{2\pi} = \frac{r^2 \theta}{2} = \frac{1}{2} r^2 \theta.}$$

Note, to use this formula, the measure of the central angle must be given in radians.

Example 8: A sector has radius 9 and central angle measuring $\frac{2\pi}{3}$ radians. Find the area of the sector.

$$A = \frac{1}{2} \cdot r^2 \cdot \theta = \frac{1}{2} \cdot 9^2 \cdot \frac{2\pi}{3} = 27\pi$$

Example 9: A sector has central angle measuring $\frac{5\pi}{3}$ radians. The area of the sector is 500π square units. Find the radius.

$$A = 500\pi \quad \theta = \frac{5\pi}{3} \quad r = ?$$

$$A = \frac{1}{2} r^2 \theta$$

$$\frac{3}{8\pi} \cdot 2 \cdot \frac{100}{500} \pi = \frac{1}{2} \cdot r^2 \cdot \frac{5\pi}{3} \cdot \frac{2}{5\pi}$$

$$600 = r^2 \Rightarrow r = \pm \sqrt{600} \quad (r > 0)$$

$$r = 10\sqrt{6}$$

$$600 = 6 \cdot 100$$

Must convert!

Example 10: Find the perimeter of a sector with central angle 60° and radius 3 in.



$$\theta = 60^\circ = \frac{\pi}{3} \text{ radians}$$

$$s = r \cdot \theta = 3 \cdot \frac{\pi}{3} = \pi \text{ rad.}$$

$$P = 2r + s = 2 \cdot 3 + \pi = 6 + \pi \text{ radians.}$$

POPPER FOR SECTION 4.2:

Question#2: Find the area of a sector that has radius 12 inches and central angle $\frac{\pi}{6}$.

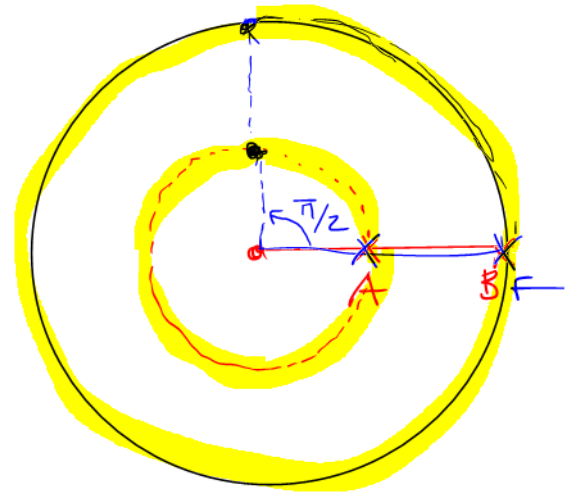
Choices on the video

Angular and Linear Velocity

Suppose you are riding on a merry-go-round.



ex: 1 sec : $\pi/2$ rad.
angular speed: $\pi/2$ rad/sec.



The ride travels in a circular motion, and the horses usually move up and down.

Some of the horses are right along the edge of the merry-go-round, and some are closer to the center. If you are on one of the horses at the edge, you will travel farther than someone who is on a horse near the center. But the length of time that both people will be on the ride is the same. If you were on the edge, not only did you travel farther, you also traveled faster.

However, everyone on the merry-go-round travels through the same number of degrees (or radians).

There are two quantities we can measure from this, angular velocity and linear velocity.

The **angular velocity** of a point on a rotating object is the number of degrees (or radians or revolutions) per unit of time through which the point turns. This will be the same for all points on the rotating object.

The **linear velocity** of a point on the rotating object is the distance per unit of time that the point travels along its circular path. This distance will depend on how far the point is from the axis of rotation (the center of the merry-go-round).

distance / time

We let the Greek letter ω represent angular velocity. Using the definition above,

$$\omega = \frac{\theta}{t}$$

We denote linear velocity by v . Using the definition above, $v = \frac{a}{t}$, where a is the arclength.

The relationship between these two quantities is given by $v = r\omega$, where r is the radius.

$$\omega = \frac{\theta}{t} \qquad v = \frac{a}{t} \qquad v = r\omega$$

$\theta = \frac{s}{t} = \frac{r \cdot \theta}{t} = r \cdot \frac{\theta}{t} = r \cdot \omega$

radius *angular*

Example 11: If the speed of a revolving gear is 25 rpm, ← rotations per minute

a. Find the number of degrees per minute through which the gear turns.

$$\omega = 25 \cdot 360 \text{ degrees/min}$$

$$\omega = 9000 \text{ deg./min}$$

1 rotation = 360°
 25 rotations / 1 min.
 ↓
 25 · 360° / 1 min

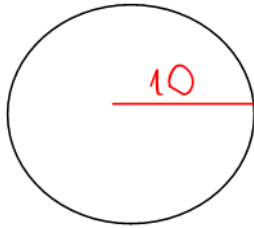
b. Find the number of radians per minute through which the gear turns.

$$\omega = 50\pi \text{ rad./min}$$

1 rot : 2π
 25 rot : 25 × 2π
 = 50π radians

Example 12: A car has wheels with a 10 inch radius. If each wheel's rate of turn is 4 revolutions per second,

a. Find the angular speed in units of radians/second.



$$1 \text{ rotation} = 2\pi \text{ rad.}$$

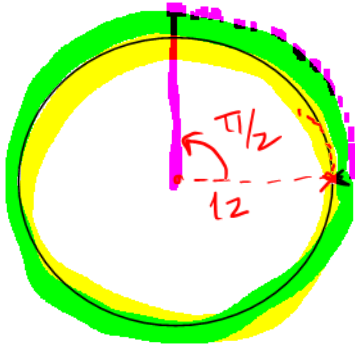
$$4 \text{ rot./sec} = 4 \cdot 2\pi \text{ rad/sec}$$

$$\omega = 4 \cdot 2\pi = 8\pi \text{ rad/sec.}$$

b. How fast (linear speed) is the car moving in units of inches/second?

$$v = r \cdot \omega = 10 \cdot 8\pi = 80\pi \text{ in./sec.}$$

Example 13: A particle is moving on a wheel with radius 12 inches with an angular speed of $\frac{\pi}{4}$ radians per second. After completing 2 full rotations, the particle traveled for 2 more seconds. What is the total distance that this particle covered?



$$1 \text{ sec} : \frac{\pi}{4} \text{ rad}$$

$$2 \text{ sec} : 2 \cdot \frac{\pi}{4} = \frac{\pi}{2} \text{ radians}$$

$$2 \text{ rotations} : 2 \cdot 2\pi r$$

$$2 \cdot 2\pi \cdot 12$$

$$= 48\pi$$

2 more seconds: covers $\frac{\pi}{2}$ radians

$$s = r \cdot \frac{\pi}{2} = 12 \cdot \frac{\pi}{2} = 6\pi$$

$$\text{Total distance: } 48\pi + 6\pi = \boxed{54\pi}$$

POPPER FOR SECTION 4.2:

Question#3: A wheel with radius 10 inches covers a distance of 40π inches in one second. What is the angular speed of this wheel?

Choices on the video