Calculus 1 Review

Limit and Continuity

Differentiation

Applications of Derivatives

Exponential and Logarithmic Functions

Integration

Must know all rules of differentiation

Rate of change, related rates, optimization.

Increasing/Decreasing Function, local/absolute extreme values, concavity

Gathering info from the graph of f, or f' or f''.

L'Hospital's rule.

Understand how integrals are defined, know all basic rules of integration, u-sub.

Intermediate Value Theorem, Extreme Value Theorem, Rolle's Theorem, Mean Value Theorem

Fundamental Theorem of Calculus

A function which is continuous on an interval does not "skip" any values, and thus its graph is an "unbroken curve." There are no "holes" in it and no "jumps." This idea is expressed coherently by the *intermediate-value theorem*.

THEOREM 2.6.1 THE INTERMEDIATE-VALUE THEOREM

If f is continuous on $[a, b]$ and K is any number between $f(a)$ and $f(b)$, then there is at least one number c in the interval (a, b) such that $f(c) = K$.

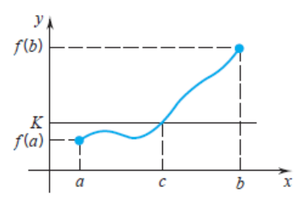


Figure 2.6.1

THEOREM 2.6.2 THE EXTREME-VALUE THEOREM

If f is continuous on a bounded closed interval $[a, b]$, then on that interval f takes on both a maximum value M and a minimum value m.

For obvious reasons, M and m are called the *extreme values* of the function.

The result is illustrated in Figure 2.6.6. The maximum value M is taken on at the point marked d, and the minimum value m is taken on at the point marked c.

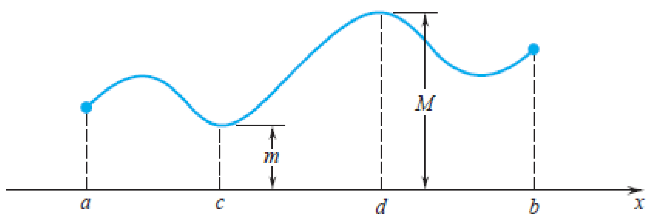


Figure 2.6.6

In Theorem 2.6.2, the full hypothesis is needed. If the interval is not bounded, the result need not hold: the cubing function $f(x) = x^3$ has no maximum on the interval $[0, \infty)$. If the interval is not closed, the result need not hold: the identity function $f(x) = x$ has no maximum and no minimum on $(0, 2)$. If the function is not continuous, the result need not hold. As an example, take the function

$$f(x) = \begin{cases} 3, & x = 1 \\ x, & 1 < x < 5 \\ 3, & x = 5. \end{cases}$$

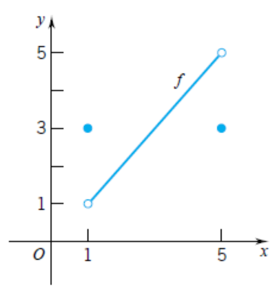


Figure 2.6.7

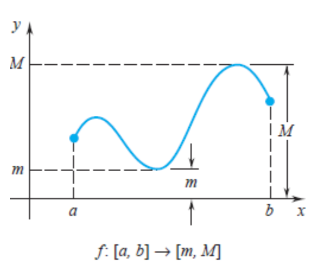
The graph is shown in Figure 2.6.7. The function is defined on $[1, 5]$, but it takes on neither a maximum value nor a minimum value. The function maps the closed interval $[1, 5]$ onto the open interval $(1, 5)$.

One final observation. From the intermediate-value theorem we know that

"continuous functions map intervals onto intervals."

Now that we have the extreme-value theorem, we know that

"continuous functions map bounded closed intervals $[a, b]$ *onto bounded closed intervals* $[m, M]$." (See Figure 2.6.8.)



$f: [a, b] \to [m, M]$

Figure 2.6.8

Of course, if f is constant, then $M = m$ and the interval $[m, M]$ collapses to a point.

A proof of the extreme-value theorem is given in Appendix B. Techniques for determining the maximum and minimum values of functions are developed in Chapter 4. These techniques require an understanding of "differentiation," the subject to which we devote Chapter 3.

Definition of derivative:

Definition: Given a function f, the **derivative of** f is the function f' defined as:

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$, provided the limit exists.

The domain of $f'(x)$ is the set of all points where the defining limit exists, that is, all x for which f is **differentiable**.

Equation of tangent line at (a,f(a)) : $y - f(a) = f'(a)(x - a)$

Rules of differentiation:

$$f(x) = k \in \mathbb{R} \Rightarrow f'(x) = 0 \qquad f(x) = e^x \Rightarrow f'(x) = e^x$$

$$f(x) = x \Rightarrow f'(x) = 1 \qquad f(x) = a^x \Rightarrow f'(x) = a^x \ln a$$

$$f(x) = x^k \Rightarrow f'(x) = kx^{k-1} \qquad f(x) = \sin x \Rightarrow f'(x) = \cos x$$

$$f(x) = \frac{1}{x} \Rightarrow f'(x) = -\frac{1}{x^2} \qquad f(x) = \cos x \Rightarrow f'(x) = -\sin x$$

$$f(x) = \sqrt{x} \Rightarrow f'(x) = \frac{1}{2\sqrt{x}} \qquad f(x) = \tan x \Rightarrow f'(x) = \sec^2 x = 1 + \tan^2 x$$

$$f(x) = \ln x \Rightarrow f'(x) = \frac{1}{x} \qquad f(x) = \arcsin x \Rightarrow f'(x) = \frac{1}{\sqrt{1-x^2}}$$

$$f(x) = \log_a x \Rightarrow f'(x) = \frac{1}{x \ln a} \qquad f(x) = \arctan x \Rightarrow f'(x) = \frac{1}{1+x^2}$$

$$\left(uv\right)' = u'v + uv'$$

$$\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}$$

$$(f \circ g)'(x) = f'(g(x))g'(x)$$

Newton's Method

Goal: To approximate a solution to $f(x)=0$ **(a root or a zero of the function f(x)).**

Start with a guess; x_0

If $f'(x_0)\neq 0$**, then next guess is:** $x_1 = x_0 - \dfrac{f(x_0)}{f'(x_0)}$

In general: $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$

THEOREM 4.1.3 ROLLE'S THEOREM

Suppose that f is differentiable on the open interval (a, b) and continuous on the closed interval $[a, b]$. If $f(a)$ and $f(b)$ are both 0, then there is at least one number c in (a, b) for which

$$f'(c) = 0.$$

Remark Rolle's theorem is sometimes formulated as follows:

Suppose that g is differentiable on the open interval (a, b) and continuous on the closed interval $[a, b]$. If $g(a) = g(b)$, then there is at least one number c in (a, b) for which

$$g'(c) = 0.$$

THEOREM 4.1.1 THE MEAN-VALUE THEOREM

If f is differentiable on the open interval (a, b) and continuous on the closed interval $[a, b]$, then there is at least one number c in (a, b) for which

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Theorem: Suppose that f is a differentiable function such that $f(a)=b$ with $f'(a)\neq 0$, then $a=f^{-1}(b)$ and:

$$(f^{-1})'(b)=\frac{1}{f'(a)}.$$

L'Hospital's Rule

Let f, g be two functions differentiable on an open interval I that contains c and suppose that $g'(x) \neq 0$ on I (except possibly at c).

If $\lim_{x \to c} f(x) = 0$ and $\lim_{x \to c} g(x) = 0$, then

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{f'(x)}{g'(x)},$$

provided that the limit on the right hand side exists.

On the other hand, if $\frac{f'(x)}{g'(x)} \to \pm\infty$ as $x \to c$, then $\frac{f(x)}{g(x)} \to \pm\infty$ as well.

L'Hospital's Rule (∞ / ∞)

Let f, g be two functions differentiable on an open interval I that contains c and suppose that $g'(x) \neq 0$ on I (except possibly at c).

If $f(x) \to \pm\infty$ and $g(x) \to \pm\infty$, then

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{f'(x)}{g'(x)},$$

provided that the limit on the right hand side exists.

On the other hand, if $\frac{f'(x)}{g'(x)} \to \pm\infty$ as $x \to c$, then $\frac{f(x)}{g(x)} \to \pm\infty$ as well.

Theorem: Fundamental Theorem of Calculus Part 1

If f is a continuous function over the interval $[a,b]$, then the function

$$F(x)=\int_a^x f(t)\ dt$$

is continuous on $[a,b]$ and differentiable on (a,b). Moreover,

$$F'(x)=\frac{d}{dx}\int_a^x f(t)\ dt=f(x), \text{ for all } x \text{ in } (a,b).$$

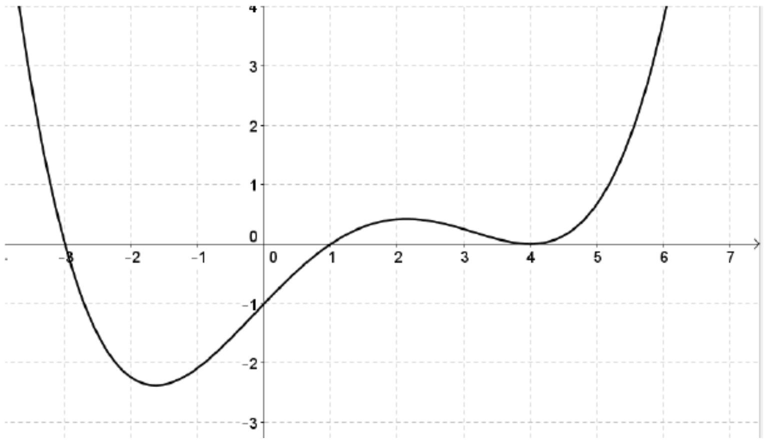
Theorem: Fundamental Theorem of Calculus Part 2

Let f be a continuous function over the interval $[a,b]$. If G is any antiderivative for f over the interval $[a,b]$, then

$$\int_a^b f(t)\ dt=G(b)-G(a).$$

Gathering information from the derivative function:

Example: Given the graph of $f'(x)$.



Exponential growth/decay

$P(t) = P_0 e^{kt}$

Half life: $kT = -\ln(2)$

Doubling time: $kT = \ln(2)$

We will see more of this topic in Differential Equations.

Word problems: If a function changes at a rate proportional to the original function, it is an exponential function.

$$\frac{df}{dt} = k \bullet f(t) \rightarrow f(t) = C \bullet e^{kt}$$

Riemann Sums

Recall – left endpoint, right end point, midpoint, Trapezoid approaches, Upper sum, Lower sum. You need to know how to put them in order depending on whether the function increases/decreases.

Example: Given the definite integral, how do these sums compare with it?

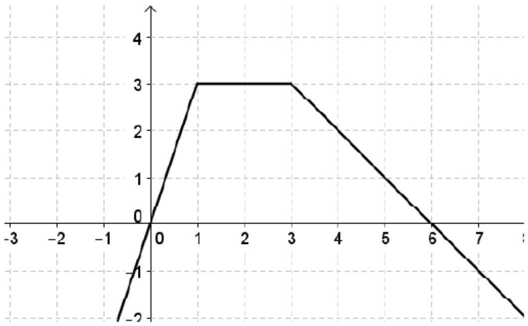
Know the definition of a definite integral.

Note: For a positive function, definite integral gives the area under the curve.

If the formula of a function is not given, but the graph is given, you can use the area under that function to find the definite integral.

Example: The graph given below belongs to $f(x)$.

$$\int_0^6 f(x)dx = ?$$



$$\int_0^2 \left(f'(x) + f(x) \right) dx = ?$$

Basic rules of integration

TABLE OF INTEGRALS

$$\int x^r \, dx = \frac{x^{r+1}}{r+1} + C \, ; \; r \neq -1$$

$$\int \frac{1}{x} \, dx = ln\left|x\right| + C$$

$$\int sin\,x \, dx = -cos\,x + C$$

$$\int cos\,x \, dx = sin\,x + C$$

$$\int sec^2 x \, dx = tan\,x + C$$

$$\int csc^2 x \, dx = -cot\,x + C$$

$$\int sec\,x\,tan\,x \, dx = sec\,x + C$$

$$\int csc\,x\,cot\,x \, dx = -csc\,x + C$$

$$\int e^x \, dx = e^x + C$$

$$\int a^x \, dx = \frac{a^x}{ln\,a} + C \; ; \quad a > 0, \, a \neq 1.$$

$$\int sinh\,x \, dx = cosh\,x + C$$

$$\int cosh\,x \, dx = sinh\,x + C$$

$$\int \frac{1}{\sqrt{1-x^2}} \, dx = arcsin\,x + C$$

$$\int \frac{1}{1+x^2} \, dx = arctan\,x + C$$

The last two formulas with u-sub:

$$\int \frac{1}{\sqrt{a^2-u^2}} \, du = arcsin\left(\frac{u}{a}\right) + C$$

$$\int \frac{1}{a^2+u^2} \, dx = \frac{1}{a} arctan\left(\frac{u}{a}\right) + C$$

TABLE OF INTEGRALS

$$\int u^n \, du = \frac{u^{n+1}}{n+1} + C, \text{ for } n \neq -1.$$

$$\int \frac{1}{u} \, du = ln|u| + C \quad \text{and} \quad \int \frac{g'(x)}{g(x)} dx = ln|g(x)| + C$$

$$\int tan\, x \, dx = -ln|cos\, x| + C = ln|sec\, x| + C$$

$$\int sec\, x \, dx = ln|sec\, x + tan\, x| + C$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} \, dx = arcsin\left(\frac{x}{a}\right) + C \qquad \int \frac{1}{\sqrt{a^2 - u^2}} \, du = arcsin\left(\frac{u}{a}\right) + C$$

$$\int \frac{1}{a^2 + x^2} \, dx = \frac{1}{a} arctan\left(\frac{x}{a}\right) + C \qquad \int \frac{1}{a^2 + u^2} \, du = \frac{1}{a} arctan\left(\frac{u}{a}\right) + C$$

Must know:

Intermediate Value Theorem

Mean Value Theorem

Implicit Differentiation

L'Hospital's rule

Related Rates

Optimization Problems

Integration

EXAMPLES

Example: Approximate a zero of the function $f(x) = x^2 - 7$ starting with an initial guess of x=3 using Newton's method with 1 iteration.

Example:

In a pure radioactive decay model the rate of change in mass, M, satisfies the differential equation

$$\frac{dM}{dt} = -\frac{M}{10}$$

If the initial mass is M_0, determine the mass after 20 units of time have passed in terms of M_0.

A. $\frac{1}{2}M_0$

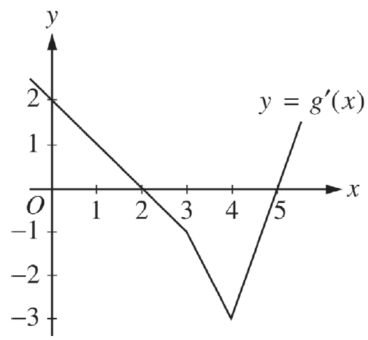
B. $\frac{1}{4}M_0$

C. $\frac{M_0}{2e}$

D. $\frac{M_0}{e}$

E. $\frac{M_0}{e^2}$

Example:

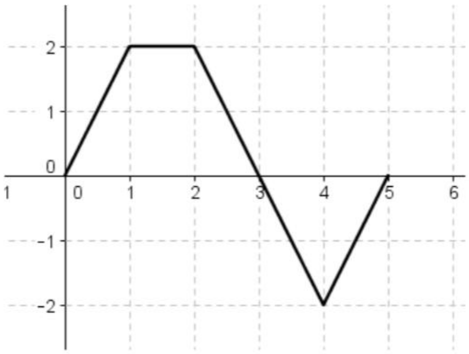


Let g be a function whose derivative g' is continuous and has the graph shown above. Which of the following values of g is largest?

(A) $g(1)$ (B) $g(2)$ (C) $g(3)$ (D) $g(4)$ (E) $g(5)$

Example:

The graph given below belongs to $f'(x)$.



Given: $f(0)=3$, find $f(4)$.

Example: Find the equation of the tangent line to $6x^2 + 2xy^2 - 5y = 4$ **at (1,2).**

Example: Find the equation of the normal line to $x^3y - 5y^2 = 3$ at (2,1).

Example: Given a rope which is 24 meters long, cut the rope and use one of the pieces to make a square and the other piece to make a circle. How should you cut the rope in order to minimize the total area? What's the area of the square in this case?

Example: $F(x) = \int_{1}^{5x} \frac{1}{t+1}\,dt$; $F'(x) = ?$

Example: $F(x) = \int_{3x}^{4x^2} \sin(2t)dt$; $F'(x) = ?$

Example: Evaluate: $\lim_{x \to 0} \left(\frac{1}{x} \int_0^{5x} \sin(2t)\,dt \right)$

Example: Evaluate: $$\lim_{x\to 0}\left(\frac{\displaystyle\int_{-x}^{0}\sin(t^3)\,dt}{x^4}\right)$$

Example: Let $f(x) = \left(\ln(x)\right)^{1/x}$; find $\lim\limits_{x\to\infty} f(x)$

Exercise:

Let $f(x) = \left(\sin(x)\right)^x$; find $\lim_{x \to 0} f(x)$

Related Rates:

s(t):position function

v(t): velocity function

a(t): acceleration function

$v(t) = s'(t), \qquad a(t) = v'(t) = s''(t)$

Example: Given velocity function $v(t) = 4t^2 - t$**, find the acceleration at time t=2.**

Example: Given acceleration function $a(t) = 4t^3 + 2t$**, if initial velocity is 5, find the velocity at time t=2.**

Note: $speed = |velocity|$

Total distance covered over [a,b] = $\int_a^b speed\ dt = \int_a^b |v(t)|\ dt$

Exercise: Make sure you can compute integrals such as these:

$$\int \frac{4x}{9+25x^2}dx$$

$$\int \frac{4}{9+25x^2}dx$$

$$\int \frac{5x+1}{16+x^2}dx$$

$$\int \frac{x^2+20}{16+x^2}dx$$

$$\int \frac{7}{\sqrt{25-x^2}}dx$$

$$\int_0^4 \frac{8x}{\sqrt{25-x^2}}dx$$

$$\int_0^{\pi} \frac{\sin(2x)}{1+\cos^2(2x)}dx$$

$$\int_0^{\pi/8} \frac{\cos(4x)}{1+\sin(4x)}dx$$