Lesson 1 – Prerequisites

1. Identifying Polynomials

To begin with, let’s review the definition of a polynomial function.

A **polynomial** is the sum and/or difference of terms that contain variables and/or real constants, with variables raised to whole number (0, 1, 2, 3, …) powers.

Example 1: Which of the following are polynomial functions?

<table>
<thead>
<tr>
<th>Expression</th>
<th>Polynomial Test</th>
<th>Result</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f(x) = -2x^5 + 0.5x + \frac{3}{2}$</td>
<td>yes</td>
<td></td>
</tr>
<tr>
<td>$f(x) = \sqrt[3]{3x^3 - 2x^3}$</td>
<td>no</td>
<td></td>
</tr>
<tr>
<td>$f(x) = \left(\frac{1}{x}\right)^{-10}x$</td>
<td>no</td>
<td></td>
</tr>
<tr>
<td>$f(x) = x^3 - 2x^2 + \frac{3}{2}x - \sqrt{5}$</td>
<td>yes</td>
<td></td>
</tr>
<tr>
<td>$f(x) = 5x^{0.5} - 0.5x^{\frac{1}{6}}$</td>
<td>no</td>
<td></td>
</tr>
<tr>
<td>$f(x) = x^6 - e$</td>
<td>no</td>
<td></td>
</tr>
<tr>
<td>$f(x) = \ln(x^2 - 3)$</td>
<td>no</td>
<td></td>
</tr>
<tr>
<td>$f(x) = 3e^{3x} - 1$</td>
<td>no</td>
<td></td>
</tr>
<tr>
<td>$f(x) = -5$</td>
<td>yes</td>
<td></td>
</tr>
</tbody>
</table>

2. Domain

The **domain** of a function is the set of all values of the independent variable(s) for which a function is defined, i.e. yields real-valued results.

Example 2: Find the domain of each of the following functions.

<table>
<thead>
<tr>
<th>Expression</th>
<th>Domain Test</th>
<th>Result</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f(x) = x^2 + 4$</td>
<td>All real numbers</td>
<td>$(-\infty, \infty)$</td>
</tr>
<tr>
<td>$f(x) = 2e^x + 3$</td>
<td>(-09, 09)</td>
<td></td>
</tr>
<tr>
<td>$f(x) = x^{\frac{3}{5}}$</td>
<td>$(-\infty, \infty)$</td>
<td>odd root</td>
</tr>
<tr>
<td>$f(x) = \sqrt{-2x + 1}$</td>
<td>even root</td>
<td></td>
</tr>
</tbody>
</table>

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e. \[ f(x) = \frac{x^2 + 7x + 12}{x^2 - 2x - 15} \]

\[ x^2 - 2x - 15 \neq 0 \]
\[ (x-5)(x+3) \neq 0 \]
\[ x \neq 5, \ x \neq -3 \]
\[ (-\infty, -3) \cup (-3, 5) \cup (5, \infty) \]

Example 3: Given the graph of a function below, give the domain.

a.

\[ (-\infty, 1) \cup (1, \infty) \]

b.

\[ (-\infty, 1) \cup (1, 2) \cup (2, \infty) \]
3. Multiplying, Solving and Evaluating Functions

Example 4: Multiply and Simplify.

\[-2x^2 + x - (x-1)(x+5)\]

\[= -2x^2 + x - (x^2 + 5x - x - 5)\]

\[= -2x^2 + x - x^2 - 4x + 5\]

\[= -3x^2 - 3x + 5\]

Example 5: Find the roots of each function below.

a. \(f(x) = 6x^3 - 22x^2 - 8x\).

\[2x(3x^2 - 11x - 4) = 0\]

\[2x(3x + 1)(x - 4) = 0\]

\[2x = 0, \quad 3x + 1 = 0, \quad x - 4 = 0\]

\[x = 0, \quad x = -\frac{1}{3}, \quad x = 4\]

b. \(f(x) = 81x^2 - 11\).

\[81x^2 = 11\]

\[x^2 = \frac{11}{81}\]

\[x = \pm \frac{\sqrt{11}}{9}\]

Example 6: Find \(f(-2)\) for each function below, if possible.

a. \(f(x) = x^3 + 10x\)

\[f(-2) = (-2)^3 + 10(-2)\]

\[= -8 - 20\]

\[= -28\]

b. \(f(x) = \frac{x-10}{x^2-4}\)

\[f(-2) = \frac{-2-10}{(-2)^2-4}\]

\[= \frac{-12}{0}\]

undefined

Example 7: Let \(f(x) = x^2 - 3\), calculate \(\frac{f(6) - f(1)}{6 - 1}\).

\[\frac{(6^2 - 3) - ((1)^2 - 3)}{5}\]

\[= \frac{36 - 3 - 1 + 3}{5}\]

\[= \frac{37}{5}\]

\[= 7\]
Example 8: Suppose the total cost in dollars to produce \( x \) items is given by the function \( C(x) = 0.0003x^3 + 0.14x^2 + 12x + 1400 \). Find the total cost of producing 50 items.

\[
C(50) = \$2387.50
\]

4. Analyzing Graphs of Polynomials

Graphs of a polynomial functions have nice, smooth curves, no sharp corners, no holes, and no asymptotes.

The **end behavior** of a polynomial function is the behavior of the polynomial to the far left and far right of the graph. If we are given only the function and not the graph, we can determine the end behavior by simply looking at its leading term (term with the highest power on the variable \( x \)).

An *even-degree* polynomial’s end behavior will be \( \uparrow \uparrow \) if its leading coefficient is positive or \( \downarrow \downarrow \) if its leading coefficient is negative.

An *odd-degree* polynomial’s end behavior will be \( \uparrow \downarrow \) if its leading coefficient is positive or \( \downarrow \uparrow \) if its leading coefficient is negative.
Example 9: Determine the end behavior of \( f(x) = -3x^5 + 2x \).

**Leading Term:** \(-3x^5\)

**Leading Coefficient:** \(-3\)

**Degree of poly:** 5th (odd)

**end behavior:** \( \uparrow \downarrow \)

Example 10: Given the following graph of a polynomial function, give the function’s degree and sign of the leading coefficient.

- **Degree:** Even
- **Sign L.C.:** Negative

Example 11: Given the following graph of a polynomial function below.

a. For which \( x \)-value(s) is the function equal to 0.

\[ f(x) = 0 \]

\[ x = \pm 1 \]

b. Give the interval(s) over which the function is negative; positive.

- **Neg:** \((-\infty, -1) \cup (1, \infty)\)
- **Pos:** \((-1, 1)\)
5. Piecewise Defined Functions

A function that is defined by two (or more) equations over a specified domain is called a piecewise function.

Example 12: Let \( f(x) = \begin{cases} \frac{3}{x} - 1, & x < 1 \\ 10e^{3x} + 6, & -1 \leq x < 3 \\ x + 5, & x \geq 3 \end{cases} \).

Find:

a. \( f(0). \)

\[
= 10e^{3(0)} + 6 \\
= 10e^0 + 6 \\
= 10(1) + 6 = 16
\]

b. \( f(\frac{1}{2}). \)

\[
= (-2)^3 - 1 \\
= -8 - 1 = -9
\]

Example 13: Use the graph above to find each of the following.

d) Find the domain and range of \( f(x). \)

\[
\text{Domain: } (-\infty, 2) \cup [3, 6] \\
\text{Range: } (-\infty, 4]
\]

c. For which \( x \)-value(s) is \( f(x) = 3. \)

\[
 \chi = \pm 1
\]
Vertical Asymptote of Rational Functions

The line \( x = a \) is a **vertical asymptote** of the graph of a function \( f \) if \( f(x) \) increases or decreases without bound as \( x \) approaches \( a \).

Example:

Given \( f(x) = \frac{1}{x} \), the line \( x = 0 \) (y-axis) is its vertical asymptote.

Finding Vertical Asymptotes and Holes Algebraically

1. Factor the numerator and denominator as much as possible.
2. Look at each factor in the denominator.
   - If a factor cancels with a factor in the numerator, then there **is a hole** where that factor equals zero.
   - If a factor does **not cancel**, then there is a **vertical asymptote** where that factor equals zero.

Example 14: Find any vertical asymptotes for \( f(x) = \frac{2x^3 + 5x^2 - 3x}{x^3 - 3x^2 - 4x} \).

\[
f(x) = \frac{x(2x^2 + 5x - 3)}{x(x^2 - 3x - 4)} = \frac{(2x - 1)(x + 3)}{(x - 4)(x + 1)}
\]

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**From Denominator:**

\[
\begin{align*}
x - 4 &= 0 &\Rightarrow x &= 4 \\
x + 1 &= 0 &\Rightarrow x &= -1
\end{align*}
\]
Horizontal Asymptote of Rational Functions

The line \( y = b \) is a **horizontal asymptote** of the graph of a function \( f \) if \( f(x) \) approaches \( b \) as \( x \) increases or decreases without bound.

Example:
Given \( f(x) = \frac{1}{x} \), the line \( y = 0 \) (x-axis) is its horizontal asymptote.

Horizontal asymptotes really have to do with what happens to the \( y \)-values as \( x \) becomes very large or very small. If the \( y \)-values approach a particular number at the far left and far right ends of the graph, then the function has a horizontal asymptote.

Note: A rational function may have several vertical asymptotes, but only at most one horizontal asymptote. Also, a graph cannot cross a vertical asymptote, but may cross a horizontal asymptote.

**Finding Horizontal Asymptotes**

Let \( f(x) = \frac{p(x)}{q(x)} \). Shorthand: degree of \( f = \deg(f) \), numerator = \( N \), denominator = \( D \)

1. If \( \deg(N) > \deg(D) \) then there is no horizontal asymptote.
2. If \( \deg(N) < \deg(D) \) then there is a horizontal asymptote and it is \( y = 0 \) (x-axis).
3. If \( \deg(N) = \deg(D) \) then there is a horizontal asymptote and it is \( y = \frac{a}{b} \), where
   - \( a \) is the leading coefficient of the numerator.
   - \( b \) is the leading coefficient of the denominator.

Example 15: Find the horizontal asymptote, if there is one, of:

a. \( f(x) = \frac{x^3 + 3x^2}{x^6 + 4x^5 - 5} \)

   \( \deg(N) \underbrace{>} \quad \deg(D) \); therefore, \underline{None}
b. \[ f(x) = \frac{x^2 + 2x + 3}{2x^2 + 6x - 1} \]
\[ \text{deg}(N) = \text{deg}(D); \text{ therefore, } y = \frac{1}{2} \]

\[ \downarrow \quad \downarrow \]
\[ 2 \quad 2 \]

c. \[ f(x) = \frac{1}{-x^3 + 5x} \]
\[ \text{deg}(N) < \text{deg}(D); \text{ therefore, } y = 0 \]
\[ \downarrow \quad \downarrow \]
\[ 0 \quad 3 \]

Now you can take Practice Test 1 (up to 20 times) then take Test 1 (up to 2 times) from anywhere online (no CASA reservation needed).