Math 2433 – Notes week 7 (part 2) – Today's material is NOT on test 2.

16.8 Cylindrical Coordinates

for double integrals → Polar coord. (2D)

"Polar" for triple integrals

$(x, y, z) \to (r, \theta, z)$

where

$x = r\cos\theta \quad r^2 = x^2 + y^2$

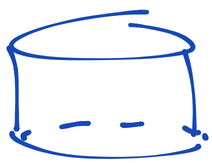
$y = r\sin\theta \quad \tan\theta = \frac{y}{x}$

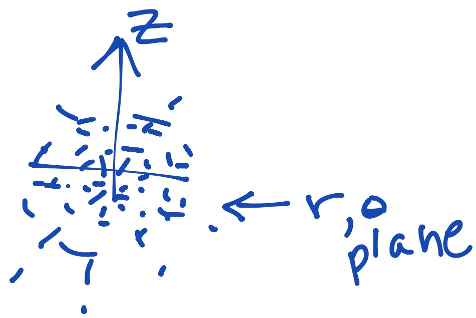
$\boxed{z = z \qquad z = z}$

$\iiint_\Omega f(x,y,z)dxdydz = \iiint_\Gamma F(r,\theta,z)rdrd\theta dz$

where

$F(r,\theta,z) = f(r\cos\theta, r\sin\theta, z)$





We use Cylindrical Coordinates when there is an axis of symmetry, the integrand involves $x^2 + y^2$, integrating over a circle or part of a circle in the xy-plane.

$$\boxed{\begin{aligned} V &= \iiint dxdydz \\ V &= \iiint r dr d\theta dz \end{aligned}}$$

Examples:

1. $\displaystyle\int_{\frac{1}{4}\pi}^{\frac{2}{3}\pi}\int_0^1\int_0^{\sqrt{1-r^2}} r\,dz\,dr\,d\theta$

$$rz\Big|_0^{\sqrt{1-r^2}} = r\sqrt{1-r^2}$$

$$\int_{\pi/4}^{2\pi/3}\underbrace{\int_0^1 r\sqrt{1-r^2}\,dr}\,d\theta$$

$u = 1 - r^2$

$du = -2r\,dr$

$$-\frac{1}{2}\cdot\frac{2}{3}(1-r^2)^{3/2}\Big|_0^1 = \frac{1}{3}$$

$$\int_{\pi/4}^{2\pi/3}\frac{1}{3}d\theta = \frac{1}{3}\theta\Big|_{\pi/4}^{2\pi/3} = \frac{2\pi}{9} - \frac{\pi}{12} = \boxed{\frac{5\pi}{36}}$$

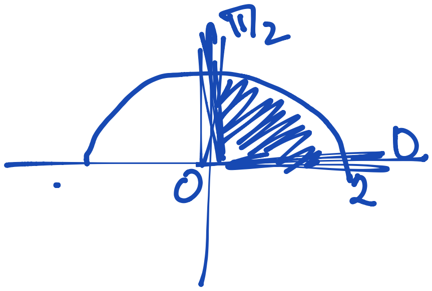
$x^2+y^2=4$ (top)

$\to r\,dr\,d\theta$

2. Evaluate the integral using cylindrical coordinates: $\iiint_T dx\,dy\,dz$ where T is the solid formed by

$\underbrace{0 \le x \le 2, 0 \le y \le \sqrt{4-x^2}}, 0 \le z \le \sqrt{16-x^2-y^2} \leftarrow$ hemisphere $x^2+y^2+z^2=16$

$-(x^2+y^2)$

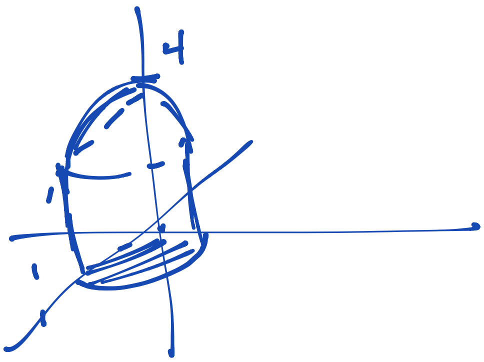


$0 \le r \le 2$

$0 \le \theta \le \pi/2$

$0 \le z \le \sqrt{16-r^2}$

$x^2+y^2=r^2$



$$\int_0^{\pi/2}\int_0^2\int_0^{\sqrt{16-r^2}} r\,dz\,dr\,d\theta$$

$$\int_0^{\pi/2}\int_0^2 \left[rz\right]_0^{\sqrt{16-r^2}} dr\,d\theta = \int_0^{\pi/2}\int_0^2 r\sqrt{16-r^2}\,dr\,d\theta$$

$$= \int_0^{\pi/2}\left[-\tfrac{1}{3}\left(16-r^2\right)^{3/2}\right]_0^2 d\theta = \int_0^{\pi/2}\left(-\tfrac{1}{3}\cdot 12^{3/2}+\tfrac{1}{3}\cdot 64\right)d\theta$$

$$= \boxed{-4\sqrt{3}\,\pi + \frac{32\pi}{3}}$$

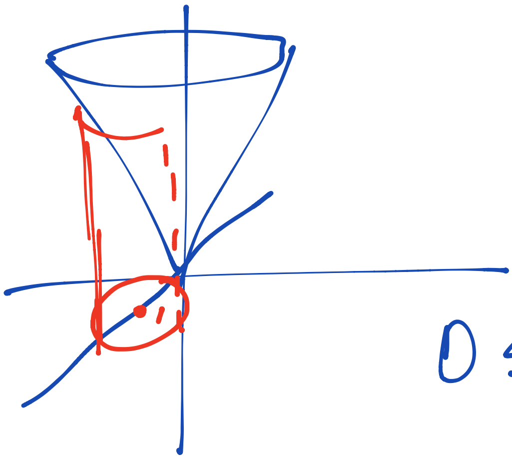
above $z = \sqrt{x^2+y^2} = \sqrt{r^2}$

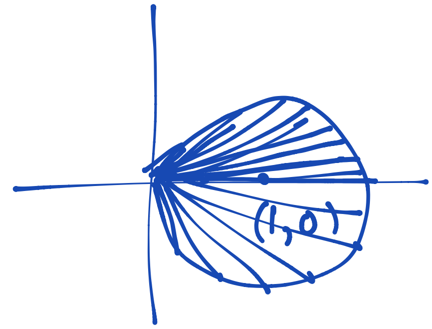
$z = 0$

3. Find the volume of the solid bounded above by the cone $z^2 = x^2 + y^2$, below by the xy-plane, and on the sides by the cylinder $x^2 + y^2 = 2x$.

$r^2 = 2r\cos\theta$

$r = 2\cos\theta$





$0 \le z \le r$

$-\frac{\pi}{2} \le \theta \le \pi/2$

$0 \le r \le 2\cos\theta$

$$V = \iiint r\,dr\,d\theta\,dz$$

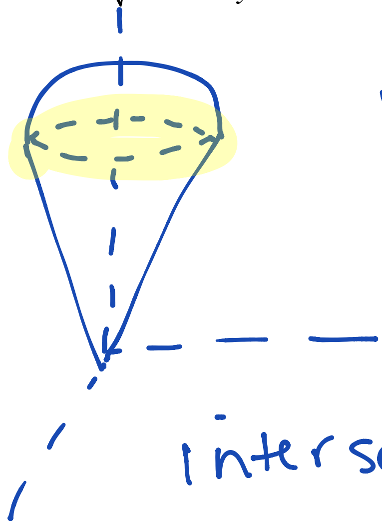
$$\int_{-\pi/2}^{\pi/2}\int_0^{2\cos\theta}\underbrace{\int_0^r r\,dz}_{rz|_0^r = r^2}\,dr\,d\theta \leftarrow \text{formula}$$

$$\int_{-\pi/2}^{\pi/2}\int_0^{2\cos\theta} r^2\,dr\,d\theta = \int_{-\pi/2}^{\pi/2} \frac{r^3}{3}\Big|_0^{2\cos\theta} d\theta = \int_{-\pi/2}^{\pi/2} \frac{8}{3}\cos^3\theta\,d\theta$$

$$\frac{8}{3}\int_{-\pi/2}^{\pi/2} \underbrace{(1-\sin^2\theta)}_{u=\sin\theta}\underbrace{\cos\theta\,d\theta}_{du} = \frac{8}{3}\left(\sin\theta - \frac{1}{3}\sin^3\theta\right)\Big|_{-\pi/2}^{\pi/2}$$

$$= \boxed{32/9}$$

4. Set up the integral to find the volume of the solid bounded above by the hemisphere $z=\sqrt{4-x^2-y^2}$ and below by the cone $z=\sqrt{3x^2+3y^2}$ using cylindrical coordinates



$$\sqrt{3(x^2+y^2)} \le z \le \sqrt{4-(x^2+y^2)}$$

$$r\sqrt{3} \le z \le \sqrt{4-r^2}$$

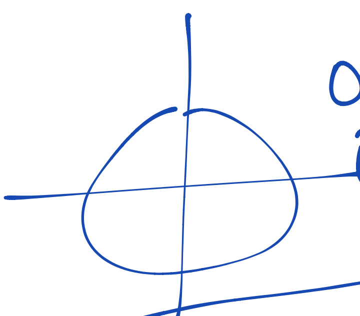
intersection $r\sqrt{3} = \sqrt{4-r^2}$

$$3r^2 = 4-r^2$$

$$4r^2 = 4$$

$$r^2 = 1 \Rightarrow r = 1$$

(circle @ origin w/ $r=1$)



$0 \le \theta \le 2\pi$

$0 \le r \le 1$

$$\boxed{V = \int_0^{2\pi}\int_0^1\int_{r\sqrt{3}}^{\sqrt{4-r^2}} r\,dz\,dr\,d\theta}$$

Last Thurs. we did this:

$$\boxed{V = \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{\sqrt{3x^2+3y^2}}^{\sqrt{4-x^2-y^2}} dz\,dy\,dx}$$

rectangular coordinates

$z=0$

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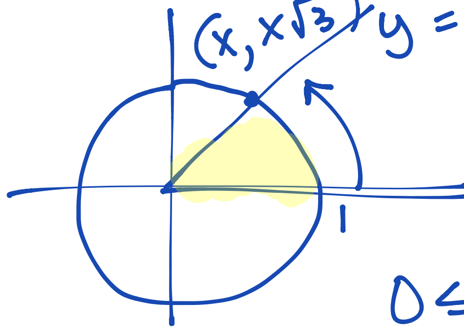
6. A solid is bounded above by the surface $z = e^{-(x^2+y^2)}$, below by the xy-plane, and on the sides by the planes $y = 0$, $y = x\sqrt{3}$, and the cylinder $x^2 + y^2 = 1$. A triple integral that gives the volume of the solid is:

$(x, x\sqrt{3})$ $y = x\sqrt{3}$



$$a. \int_0^{\frac{\pi}{6}} \int_0^1 \int_0^{e^{-r^2}} r\,dz\,dr\,d\theta$$

$0 \le r \le 1$

$\le \theta \le$

$$b. \int_0^1 \int_{\frac{y}{\sqrt{3}}}^{\sqrt{1-y^2}} \int_0^{e^{-x^2-y^2}} dz\,dx\,dy$$

$$c. \int_0^1 \int_0^{x\sqrt{3}} \int_0^{e^{-x^2-y^2}} dz\,dy\,dx$$

$$d. \int_0^{\frac{\pi}{3}} \int_0^1 \int_0^{e^{-r^2}} r\,dz\,dr\,d\theta$$

e. none of the above

$\boxed{?}$ $\boxed{2\sin(r^2)\, r\,dr\,d\theta}$ dz

7. Give the integral using cylindrical coordinates to find:

$$\iiint 2\sin\left(x^2+y^2\right) dx\,dy\,dz$$

Where T: $-1 \le x \le 1$, $0 \le y \le \sqrt{1-x^2}$, $0 \le z \le 2$.

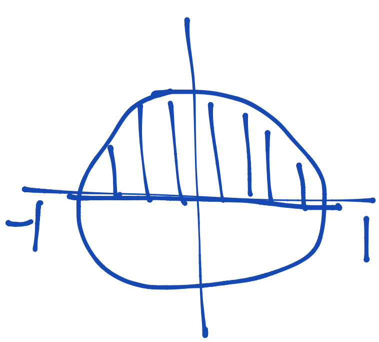
a) $\int_0^{\pi}\int_0^1\int_0^2 2\sin\left(r^2\right) r\,dz\,dr\,d\theta$

b) $\int_0^{\pi/2}\int_0^1\int_0^2 2\sin\left(r^2\right) r\,dz\,dr\,d\theta$

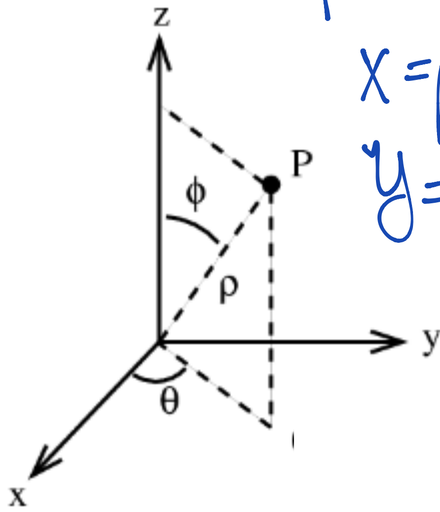
c) $\int_0^{2\pi}\int_0^1\int_0^2 2\sin\left(r^2\right) r\,dz\,dr\,d\theta$

~~d)~~ $\int_0^{\pi}\int_0^1\int_0^2 2\sin\left(r^2\right) dz\,dr\,d\theta$

?



ρ = radius of sphere



$x = \rho \sin\phi \cos\theta$
$y = \rho \sin\phi \sin\theta$
$z = \rho \cos\phi$

16.9 Spherical Coordinates

$(x, y, z) \to (\rho, \theta, \phi)$ where

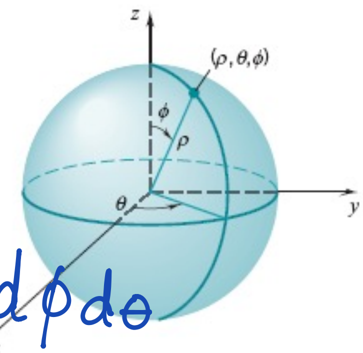
$\rho^2 = x^2 + y^2 + z^2 = r^2 + z^2$

$x = \rho \sin\phi \cos\theta$

$y = \rho \sin\phi \sin\theta$

$(notice: r = \rho \sin\phi)$

$z = \rho \cos\phi \quad (so, \cos\phi = \frac{z}{\rho})$



http://www.math.uri.edu/~bkaskosz/flashmo/tools/sphcoords/

Triple Integrals

$$\iiint_T f(x,y,z)\,dx\,dy\,dz = \iiint_S f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)\,\rho^2 \sin\phi \; d\rho\, d\phi\, d\theta$$

NOTE: $dxdydz \to \rho^2 \sin\phi\, d\rho\, d\theta\, d\phi$
Use when there is spherical symmetry and you are integrating over a sphere or part of a sphere.
Integrand will involve $x^2 + y^2 + z^2$

Examples:

1. Find the rectangular coordinates of the point with spherical coordinates $(\rho, \theta, \phi) = \left(3, \frac{\pi}{3}, \frac{3\pi}{4}\right)$

$\rho = 3 \quad \phi = \frac{3\pi}{4}$
$\theta = \pi/3$

$x = \rho \sin\phi \cos\theta = 3 \sin\frac{3\pi}{4} \cos\frac{\pi}{3} = \frac{3\sqrt{2}}{4}$
$y = \rho \sin\phi \sin\theta = 3 \sin 3\pi/4 \sin \pi/3 = \frac{3\sqrt{6}}{4}$
$z = \rho \cos\phi = 3 \cos\frac{3\pi}{4} = \frac{-3\sqrt{2}}{2}$

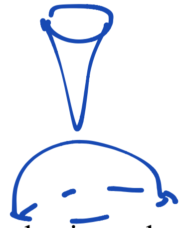
2. Find the spherical coordinates, (ρ, θ, ϕ), of the point with cylindrical coordinates $(r, \theta, z) = \left(\frac{\sqrt{2}}{2}, \frac{\pi}{4}, \frac{1}{2}\right)$

$r = \sqrt{2}/2$
$\theta = \pi/4$
$z = 1/2$

$\rho^2 = r^2 + z^2 = \frac{1}{2} + \frac{1}{4} = 3/4 \Rightarrow \rho = \sqrt{3}/2$
$\theta = \pi/4$
ϕ : $z = \rho \cos\phi$
$1/2 = \sqrt{3}/2 \cos\phi \to \cos\phi = \frac{1}{\sqrt{3}}$

$z=\sqrt{x^2+y^2}$

$z=\sqrt{\rho^2-x^2-y^2}$



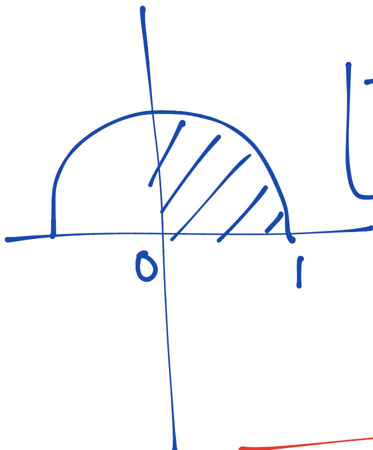
$\phi=\cos^{-1}\left(\frac{1}{\sqrt{3}}\right)$

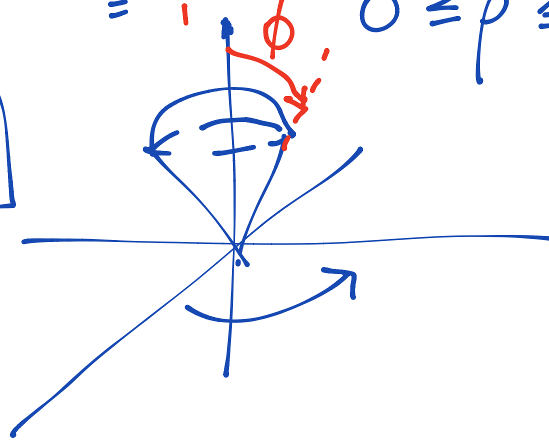
3. Evaluate the integral using spherical coordinates: $\iiint_T dx\,dy\,dz$ where T is the solid formed by $0\le x\le 1, 0\le y\le\sqrt{1-x^2}, \sqrt{x^2+y^2}\le z\le\sqrt{2-x^2-y^2}$

$0\le\rho\le\sqrt{2}$

$0\le r\le 1$

$\boxed{0\le\theta\le\pi/2}$





$\sqrt{r^2}=\sqrt{2-r^2}$

$r^2=2-r^2$

$2r^2=2$

$r^2=1$

$r=1\Rightarrow x^2+y^2=1\Rightarrow$

$\left.\begin{array}{l} z=\sqrt{x^2+y^2} \\ z=\sqrt{2-x^2-y^2}\end{array}\right\}\Rightarrow z=1$

$z=\rho\cos\phi$

$1=\sqrt{2}\cos\phi$

$1/\sqrt{2}=\cos\phi\Rightarrow\phi=\pi/4$

$\Rightarrow 0\le\phi\le\pi/4$

$$\boxed{\int_0^{\pi/2}\int_0^{\pi/4}\underbrace{\int_0^{\sqrt{2}}\rho^2\sin\phi\,d\rho}_{\frac{1}{3}\rho^3\sin\phi\Big|_0^{\sqrt{2}}}\,d\phi\,d\theta}$$

$$\int_0^{\pi/2}\underbrace{\int_0^{\pi/4}\frac{2\sqrt{2}}{3}\sin\phi\,d\phi}_{-\frac{2\sqrt{2}}{3}\cos\phi\Big|_0^{\pi/4}}\,d\theta$$

$-\frac{2\sqrt{2}}{3}\left(\frac{\sqrt{2}}{2}\right)+\frac{2\sqrt{2}}{3}$

$-2/3+\frac{2\sqrt{2}}{3}$

$$\int_0^{\pi/2}\left(-2/3+2\sqrt{2}/3\right)d\theta=\boxed{-\frac{\pi}{3}+\frac{\pi\sqrt{2}}{3}}$$

Set up

4. ~~Evaluate~~ the integral using spherical coordinates: $\iiint_T z\sqrt{x^2+y^2+z^2}\,dx\,dy\,dz$ where T is the solid formed by $0 \le x \le \sqrt{4-y^2}, 0 \le y \le 2, 0 \le z \le \sqrt{4-x^2-y^2}$ ← hemisphere

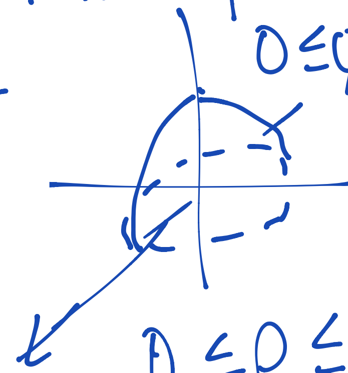
$\rho = 2$

$0 \le \phi \le \frac{\pi}{2}$

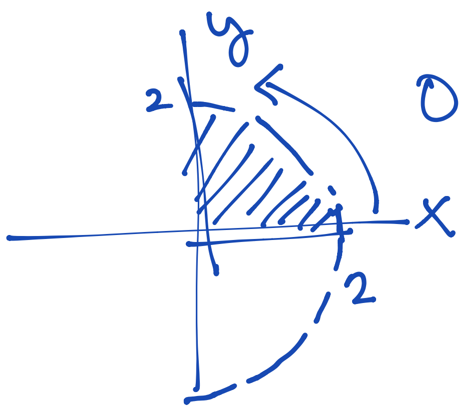
$\sqrt{x^2+y^2+z^2} = \sqrt{\rho^2} = \rho$

$z = \rho\cos\phi$

$dx\,dy\,dz \to \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta$



$0 \le \rho \le 2$



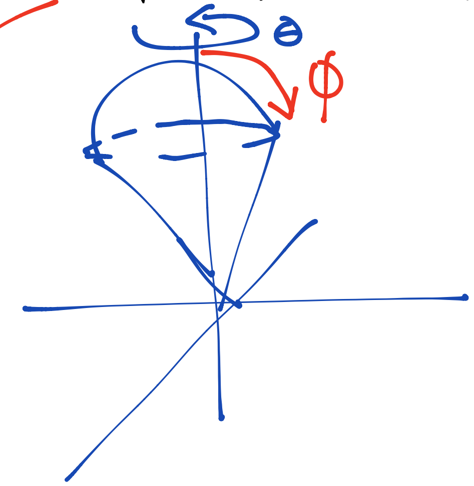
$0 \le \theta \le \pi/2$

$$\int_0^{\pi/2}\int_0^{\pi/2}\int_0^{2} \rho\cos\phi \cdot \rho \cdot \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta$$

$$= \int_0^{\pi/2}\int_0^{\pi/2}\int_0^{2} \rho^4 \cos\phi \sin\phi\, d\rho\, d\phi\, d\theta$$

$\rho = 2 \Rightarrow 0 \le \rho \le 2$

5. Set up the integral to find the volume of the solid bounded above by the hemisphere $z = \sqrt{4 - x^2 - y^2}$ and below by the cone $z = \sqrt{3x^2 + 3y^2}$ using spherical coordinates



$0 \le \theta \le 2\pi$

$0 \le \phi \le \pi/6$

Intersect

$\sqrt{4 - r^2} = \sqrt{3r^2}$

$4 - r^2 = 3r^2$

$4 = 4r^2$

$1 = r$

$z = \sqrt{4 - 1} = \sqrt{3}$

$z = \sqrt{3(1)}$ ✓

$z = \rho\cos\phi$

$\sqrt{3} = 2\cos\phi$

$\frac{\sqrt{3}}{2} = \cos\phi$

$\phi = \pi/6$

$$V = \int_0^{2\pi}\int_0^{\pi/6}\int_0^{2} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

radius of sphere

Sphere: $x^2+y^2+z^2=\rho^2$

Popper 07

8. Find the spherical coordinates of the point with cylindrical coordinates :

$$\left(\frac{1}{2}, \frac{1}{6}\pi, \frac{1}{2}\right)$$

$r = 1/2 \quad \theta = \pi/6 \quad z = \frac{1}{2}$

ρ, θ, ϕ

a) $\left(\frac{1}{4}\sqrt{3}, \frac{1}{4}, \frac{1}{2}\sqrt{2}\right)$

b) $\left(\frac{1}{2}\sqrt{2}, \frac{1}{6}\pi, \frac{1}{2}\right)$

c) $\left(\frac{1}{4}\sqrt{3}, \frac{1}{4}, \frac{1}{4}\pi\right)$

d) $\left(\frac{1}{2}\sqrt{2}, \frac{1}{6}\pi, \frac{1}{4}\pi\right)$

e) $\left(\frac{1}{4}\sqrt{3}, \frac{1}{4}, \frac{1}{2}\right)$

9. Which of the following will find the integral in spherical coordinates?

$$\iiint 2z\sqrt{x^2+y^2+z^2}\, dx\, dy\, dz$$

like ex 4

Where T: $0 \le x \le \sqrt{1-y^2}$, $0 \le y \le 1$, $0 \le z \le \sqrt{1-x^2-y^2}$

a) $\int_0^{\pi/2}\int_0^{\pi/2}\int_0^1 2\rho^4 \cos\phi \sin\phi \, d\rho\, d\theta\, d\phi$

b) $\int_0^{\pi/2}\int_0^{\pi/2}\int_0^1 2\rho^3 \cos\phi \sin\phi \, d\rho\, d\theta\, d\phi$

c) $\int_0^{\pi/2}\int_0^{\pi}\int_0^1 2\rho^4 \cos\phi \sin\phi \, d\rho\, d\theta\, d\phi$

d) $\int_0^{\pi}\int_0^{\pi/2}\int_0^1 2\rho^3 \sin\phi \, d\rho\, d\theta\, d\phi$

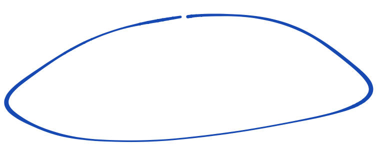
e) $\int_0^{\pi/2}\int_0^{\pi}\int_0^1 2\rho^3 \sin\phi \, d\rho\, d\theta\, d\phi$

16.10 The Jacobian; Changing Variables in Multiple Integration

So far, we have used Polar, Cylindrical and Spherical coordinates to make some of our integration problems easier.
In calc I, we used u-sub to make some integration easier. We are now going to do something similar but now we will change the *xy*-coordinates into *uv*-coordinates. To do that, we will need to transform our region too.

Example:

Transform the region Ω: $x^2+\frac{y^2}{36}=1$ into Γ using the transformation $x=\frac{1}{2}u,\ y=3v$



ellipse

$$\left(\frac{1}{2}u\right)^2+\frac{(3v)^2}{36}=1$$

$$\frac{u^2}{4}+\frac{9v^2}{36}=1$$

$$u^2+v^2=4$$

circle

To change our variables in a double integral using our transformation, we first need to find the **Jacobian**:

The Jacobian of the transformation $(x,y) \to (x(u,v), y(u,v))$ is

$$J(u,v) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix}$$

Example: Find the Jacobian of the transformation $x = 4uv,\ y = u^2 + 2v^2$

$$\frac{\partial x}{\partial u} = 4v \qquad \frac{\partial y}{\partial u} = 2u$$

$$\frac{\partial x}{\partial v} = 4u \qquad \frac{\partial y}{\partial v} = 4v$$

$$J = \begin{vmatrix} 4v & 2u \\ 4u & 4v \end{vmatrix} = \boxed{16v^2 - 8u^2}$$

Jacobian for

Now, if the Jacobian is not 0 then the area of $\Omega = \iint_\Gamma |J(u,v)|\, dudv$

absolute value of Jac.

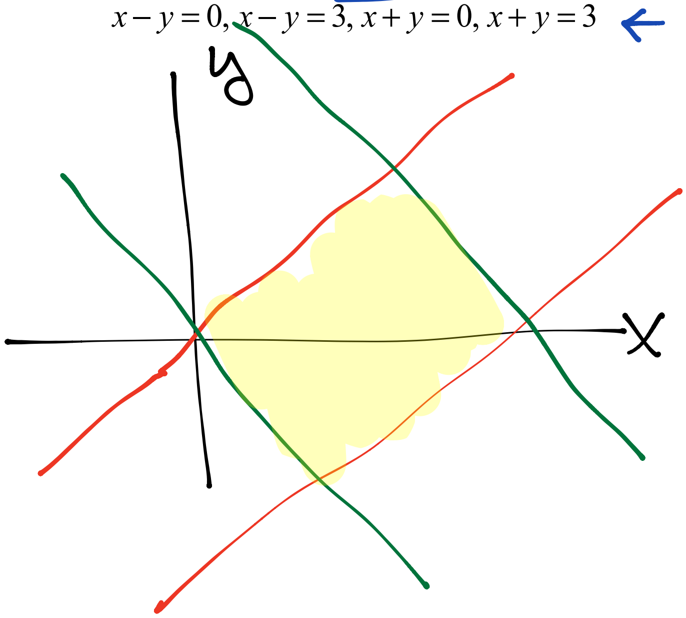
Where $\mathbf{\Gamma}$ represents

And so, $\iint_\Omega f(x,y)dxdy = \iint_\Gamma f(x(u,v), y(u,v))|J(u,v)|\, dudv$

$$dx\,dy \to |J(u,v)|\,du\,dv$$

Example: Evaluate $\boxed{\iint_\Omega (x+y)\cos(\pi(x-y))\,dx\,dy}$ where Ω is the parallelogram bounded by

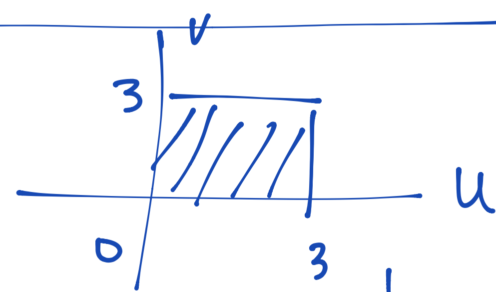
$x-y=0,\ x-y=3,\ x+y=0,\ x+y=3$ ←



$$0 \le x-y \le 3$$
$$0 \le x+y \le 3$$

$$u = x-y \qquad v = x+y$$

$$\boxed{0 \le u \le 3 \qquad 0 \le v \le 3}$$



NICE!

$$u = x-y$$
$$v = x+y$$
$$u+v = 2x$$
$$\boxed{x = \tfrac{1}{2}u + \tfrac{1}{2}v}$$

$$-u = -x+y$$
$$v = x+y$$
$$v-u = 2y$$
$$\boxed{y = \tfrac{1}{2}v - \tfrac{1}{2}u}$$

$$\frac{\partial x}{\partial u} = \frac{1}{2} \qquad \frac{\partial y}{\partial u} = -\frac{1}{2}$$

$$\frac{\partial x}{\partial v} = \frac{1}{2} \qquad \frac{\partial y}{\partial v} = \frac{1}{2}$$

$$J = \begin{vmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{vmatrix} = \frac{1}{4} - -\frac{1}{4} = \frac{1}{2}$$

$$\boxed{\int_0^3 \int_0^3 v\cos(\pi u)\left|\frac{1}{2}\right| du\,dv}$$

Another example: Evaluate:

$$\iint_\Omega xy \, dxdy = \int_1^4 \int_4^9 \frac{1}{2}\sqrt{u^2-v^2} \left| \frac{-1}{4\sqrt{u^2-v^2}} \right| du\,dv$$

$$\boxed{\int_1^4 \int_4^9 \frac{1}{8}\, du\, dv = \boxed{\frac{15}{8}}}$$

where Ω is the first-quadrant region bounded by the curves

$$\underbrace{x^2+y^2=4, \quad x^2+y^2=9,}_{u=x^2+y^2} \quad x^2-y^2=1, \quad x^2-y^2=4.$$

$v = x^2-y^2$

$4 \le u \le 9$ $\qquad$ $1 \le v \le 4$

$\int_1^4 \int_4^9 du\,dv$

$= \text{area}$

$4 \} 3$, 1, 4, $\boxed{15}$, 9, $\underbrace{}_{5}$

$$\begin{array}{l} u = x^2+y^2 \\ v = x^2-y^2 \\ \hline u+v = 2x^2 \end{array} \qquad \begin{array}{l} u = x^2+y^2 \\ -v = -x^2+y^2 \\ \hline u-v = 2y^2 \end{array}$$

$x = \sqrt{\frac{1}{2}u+\frac{1}{2}v}$ $\qquad$ $y = \sqrt{\frac{1}{2}u-\frac{1}{2}v}$

$x \cdot y = \sqrt{\frac{1}{4}u^2-\frac{1}{4}v^2}$

$= \frac{1}{2}\sqrt{u^2-v^2}$

$$\frac{\partial x}{\partial u} = \frac{1}{2}\left(\frac{1}{2}u+\frac{1}{2}v\right)^{-1/2}\left(\frac{1}{2}\right)$$

$$\frac{\partial x}{\partial v} \nearrow$$

$$J = \begin{vmatrix} \frac{1}{4\sqrt{\frac{1}{2}u+\frac{1}{2}v}} & \frac{1}{4\sqrt{\frac{1}{2}u+\frac{1}{2}v}} \\ \frac{1}{4\sqrt{\frac{1}{2}u-\frac{1}{2}v}} & \frac{-1}{4\sqrt{\frac{1}{2}u-\frac{1}{2}v}} \end{vmatrix}$$

$$\frac{\partial y}{\partial u} = \frac{1}{2}\left(\frac{1}{2}u-\frac{1}{2}v\right)^{-1/2}\left(\frac{1}{2}\right)$$

$$\frac{\partial y}{\partial v} = \frac{1}{2}\left(\frac{1}{2}u-\frac{1}{2}v\right)^{-1/2}\left(-\frac{1}{2}\right)$$

$$= \frac{-1}{16\sqrt{\frac{1}{4}u^2-\frac{1}{4}v^2}} - \frac{1}{16\sqrt{\frac{1}{4}u^2-\frac{1}{4}v^2}}$$

$$= \frac{-2}{8\sqrt{u^2-v^2}}$$

When changing variables in a triple integral we make three coordinate changes:

$$x = x(u, v, w), \qquad y = y(u, v, w), \qquad z = z(u, v, w).$$

If these functions carry a basic solid Γ onto a solid T, then, under conditions analogous to the two-dimensional case,

$$\text{volume of } T = \iiint_{\Gamma} |J(u, v, w)| \, du dv dw$$

where now the Jacobian† is a three-by-three determinant:

$$J(u, v, w) = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial y}{\partial u} & \dfrac{\partial z}{\partial u} \\ \dfrac{\partial x}{\partial v} & \dfrac{\partial y}{\partial v} & \dfrac{\partial z}{\partial v} \\ \dfrac{\partial x}{\partial w} & \dfrac{\partial y}{\partial w} & \dfrac{\partial z}{\partial w} \end{vmatrix}.$$

In this case the change of variables formula reads

$$\iiint_{T} f(x, y, z) \; dx dy dz = \iiint_{\Gamma} f(x(u, v, w), \; y(u, v, w), \; z(u, v, w)) |J(u, v, w)| \, du dv dw.$$

Popper07

10. Find the Jacobian of the transformation:

$$x = 3u\ln(v)$$
$$y = uv$$

a) $3\ln(v)\,v - \dfrac{3u^2}{v}$

b) $3u\ln(v) - 3u$

c) $3u\ln(v) + 3u$

d) $3\ln(v) - v$

e) $-3\ln(v)\,v + \dfrac{3u^2}{v}$

11. Find the Jacobian of the transformation:

$$x = 2v + w$$
$$y = 3u + w$$
$$z = u + v$$

a) 1

b) 0

c) 5

d) 10

e) -5

$\dfrac{\partial x}{\partial u} = 0 \qquad \dfrac{\partial y}{\partial u} = 3 \qquad \dfrac{\partial z}{\partial u} = 1$

$\dfrac{\partial x}{\partial v} = 2 \qquad \dfrac{\partial y}{\partial v} = 0 \qquad \dfrac{\partial z}{\partial v} = 1$

$\dfrac{\partial x}{\partial w} = 1 \qquad \dfrac{\partial y}{\partial w} = 1 \qquad \dfrac{\partial z}{\partial w} = 0$

$$J = \begin{vmatrix} 0 & 3 & 1 \\ 2 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix} \begin{matrix} 0 & 3 \\ 2 & 0 \\ 1 & 1 \end{matrix} = (0+3+2) - (0+0+0)$$

$$= 5$$