Stochastic Processes - Spring 2008

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Exercise Sheet 2, with Solutions

Do all Exercises individually.

(1) Let $X_1, X_2, \ldots, X_n, \ldots$ be i.i.d. with $\mathbb{P}(X_i = 1) = p$ and $\mathbb{P}(X_i = -1) = 1 - p = q$. Let $a, b \in \mathbb{N}$. Define $S_n = X_1 + \ldots + X_n$ and $S_0 = 0$, and let

$$T(\omega) = \inf\{n : S_n(\omega) = -a \text{ or } b\}.$$

Use a stopping time argument to compute the expected value $\mathbb{E}[T]$.
Solution. Define $Y_n = S_n - n(p-q)$ and $Z_n = (\frac{q}{p})^{S_n}$. It is straightforward to show that Y_n and Z_n are martingales. Define a stopping time

$$T = \min\{n : S_n = -a \text{ or } S_n = b\}$$

By Doob's stopping time theorem $\mathbb{E}[Z_T] = \mathbb{E}[Z_0] = 1$. Let P_a be the probability that the random walk reaches $-a$ before b. Then

$$1 = P_a \left(\frac{q}{p}\right)^{-a} + (1 - P_a)\left(\frac{q}{p}\right)^{b}$$

Solving for P_a gives

$$P_a = \left(\left(\frac{q}{p}\right)^{a} - \left(\frac{q}{p}\right)^{b+a}\right) \Big/ \left(1 - \left(\frac{q}{p}\right)^{b+a}\right)$$

Since Y_n is also a martingale $\mathbb{E}[Y_T] = 0$ so $\mathbb{E}[S_T] - (p-q)\mathbb{E}[T] = 0$. Thus

$$\mathbb{E}[T] = \frac{\mathbb{E}[S_T]}{p-q}$$
$$= \frac{b - (a+b)P_a}{p-q}$$

where P_a is as above.

(2) Let $\{X_1, X_2, \ldots\}$ be a sequence of i.i.d. real-valued random variables having finite expectation, and N be a stopping time for the discrete filtration generated by $\{X_n\}_{n=1}^{\infty}$ such that $\mathbb{E}[N] < \infty$.

(a) Why is $\mathbb{E}[X_n 1_{n\le N}] = \mathbb{E}[X_n]\mathbb{P}[\{\omega : N(\omega) \ge n\}]$ for each $n \in \mathbb{N}$?
Solution. The event $\{N \ge n\}$ is in the sigma-field $\mathcal{F}_{n-1}$, because the complement $\{N < n\} = \{N \le n-1\}$ is. Now, since X_n is independent of $\mathcal{F}_{n-1}$, the expectation factorizes.
(b) Using the previous step, prove

$$\mathbb{E}[\sum_{n=1}^{N} X_n] = \mathbb{E}[N]\mathbb{E}[X_1]\,.$$

Solution. By inserting the indicator function, we can change the limit of the series,

$$\mathbb{E}[\sum_{n=1}^{N} X_n] = \mathbb{E}[\sum_{n=1}^{\infty} X_n 1_{n\le N}]\,.$$

Exchanging the summation with the expectation gives

$$\mathbb{E}[\sum_{n=1}^{N} X_n] = \sum_{n=1}^{\infty} \mathbb{E}[X_n 1_{n\le N}] = \sum_{n=1}^{\infty} \mathbb{E}[X_n]\mathbb{P}[N \ge n]\,.$$

Using the fact that all X_n are identically distributed (so they have the same expected value) and a counting argument which shows $\sum_{n=1}^{\infty} \mathbb{P}[N \ge n] = \mathbb{E}[\sum_{n=1}^{\infty} 1_{N\ge n}] = \mathbb{E}[N]$, we obtain

$$\mathbb{E}[\sum_{n=1}^{N} X_n] = \mathbb{E}[X_1]\mathbb{E}[N]\,.$$

(3) Suppose that two candidates run for election. Candidate A obtains a votes and Candidate B obtains $b < a$ votes. Suppose $n = a + b$ is the total number of votes cast. Let S_k be the number of votes by which Candidate A is leading after k votes are counted (S_k can be positive or negative) so that $S_n = a - b$. For $0 \le k \le n-1$ define

$$X_k = \frac{S_{n-k}}{n-k}$$

(a) Show that $X_0, X_1, \dots, X_{n-1}$ forms a martingale.

Solution. Suppose $S_{n-k} = t$ i.e. $X_k = \frac{t}{n-k}$. If γ_a is the number of votes counted for Candidate A at time $n-k$ and γ_b the number counted for B then $\gamma_a + \gamma_b = n-k$ and $\gamma_a - \gamma_b = t$. Solving we get $\gamma_a = \frac{n-k+t}{2}, \gamma_b = \frac{n-k-t}{2}$. Going to X_{k+1} is the same as removing one vote from the counted votes at time $n-k$ to get the number and type of votes counted at time $n-k-1$. Thus $\mathbb{E}[S_{n-k-1}|S_{n-k}] = \frac{\gamma_a}{\gamma_a+\gamma_b}(t-1) + \frac{\gamma_b}{\gamma_a+\gamma_b}(t+1) = \frac{t(n-k-1)}{n-k}$. Thus $\mathbb{E}[X_{k+1}|X_0, \ldots, X_k] = X_k$.
(b) Let

$$T = \min\{k : X_k = 0\}$$

if such a k exists and $T = n-1$ otherwise. Show that T is a stopping time.
Solution. Let $T = \min\{n-1, k \text{ such that } X_k = 0\}$. Then we only have to show $\{T = n\} \in \mathcal{F}_n$. This is clear because the set $\{T = n\}$ is given by the intersection of the sets $\{X_k \neq 0\}$ for $k \leq n-1$ and the set $\{X_n = 0\}$, which are all in $\mathcal{F}_n$.
(c) Show that the probability that Candidate A leads throughout the count is $\frac{a-b}{a+b}$.
Solution. By Doob's stopping theorem, $\mathbb{E}[X_T] = \gamma = (a-b)/n$ where γ is the probability that Candidate A leads all the way in the count. Hence $\gamma = (a-b)/(a+b)$. Note that in the equation for $\mathbb{E}[X_T]$ we used the fact that if $S_1 = -1$ then $T < n-1$.

(4) Let T be the first time a standard Brownian motion crosses the line $l(t) = \beta t - \alpha$, $(\alpha > 0, \beta > 0)$. Determine the characteristic (moment generating) function of T and hence find the expected value of T. Hint: Use a martingale method.
Solution. Let $T = \min\{t : B_t(\omega) = \beta t - \alpha\}$. The random variable T is a stopping time as $\{T \leq t\} \in \sigma(B_s : 0 \leq s \leq t)$.

Let $V_t = e^{\lambda B_t - \frac{1}{2}\lambda^2 t}$, with a fixed parameter $\lambda \geq 0$. It is straightforward to verify that V_t is a martingale. It is not so easy that T satisfies the conditions of Doobs martingale theorem but we will assume they are satisfied and apply the stopping time theorem to yield

$$E[e^{\lambda(\beta T - \alpha) - \frac{1}{2}\lambda^2 T}] = 1$$

$$= E[e^{-\lambda\alpha + T(\lambda\beta - \frac{1}{2}\lambda^2)}]$$

Let $z = \lambda\beta - \frac{1}{2}\lambda^2$ so that $1 = e^{-\lambda\alpha}E[e^{zT}]$ which gives $E[e^{zT}] = e^{\lambda\alpha}$ where $\lambda = \beta - \sqrt{\beta^2 - 2z}$ (branch chosen to give $\lambda = z = 0$). Differentiating at $z = 0$ gives $E[T] = \alpha/\beta$.

(5) Give an expression for the probability that standard Brownian motion, starting at $x = 2$ at time $t = 0$ i.e. $B_0 = 2$, satisfies $B_t < -1$ for some $0 \leq t \leq 3$. Evaluate this expression numerically.

We consider instead a standard BM W_t starting at 0 and ask $P(W_t \leq -3)$ for some $0 \leq t \leq 3$. Now $P(W_t \leq -3) = P(-W_t \geq 3)$ for some $0 \leq t \leq 3$. But $-W_t$ is also a standard BM. Hence by reflection principle $P(-W_t \geq 3) = \frac{2}{\sqrt{6\pi}} \int_3^\infty e^{-x^2/6} dx$.

(6) Suppose that $\{B_t\}$ and $\{W_t\}$ are independent standard Brownian motions (starting at zero), and let $\rho \in [0, 1]$ be a constant. Is the process $X_t = \rho B_t + \sqrt{1 - \rho^2} W_t$ a standard Brownian motion? Justify your answer!

Solution. Since both B_t and W_t are Gaussian processes, so is X_t. Moreover, the linear combination of continuous processes is continuous and the expected value is zero. It remains to check the covariance. By the independence of W_t and B_t, we can drop the mixed terms and obtain the desired covariance function

$$\mathbb{E}[X_t X_s] = \rho^2 \mathbb{E}[B_t B_s] + (1 - \rho^2)\mathbb{E}[W_t W_s] = \min\{s, t\}\,.$$