Popper # 12

$1 \text{ rad} = \left(\frac{180}{\pi}\right)^\circ$

① Convert $\frac{\pi}{6}$ rad in degrees: $\frac{\cancel{\pi}}{6} \times \frac{180}{\cancel{\pi}} = \frac{180}{6} = 30^\circ$

A. 30° B. 60° C. 180° D. none

$1^\circ = \frac{\pi}{180}$ rad

② Convert -135° in radians: $-135^\circ = -\overset{3}{\cancel{135}} \times \frac{\pi}{\underset{4}{\cancel{180}}} = -\frac{3\pi}{4}$

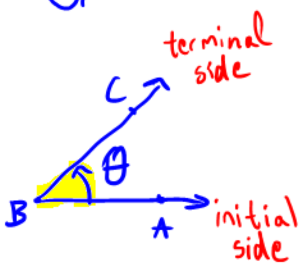
A. $\frac{3\pi}{4}$ B. $-\frac{3\pi}{4}$ C. $\frac{-\pi}{2}$ D. none

③ Mark $\boxed{A}$

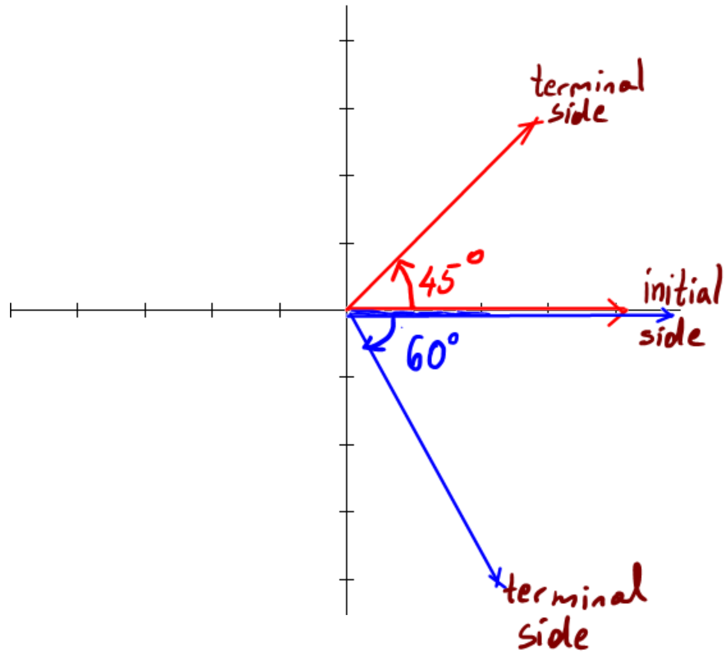
To be continued on Wednesday, 10/07

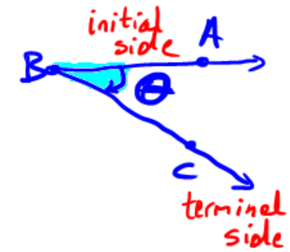
Math 1330 - Section 4.3
Unit Circle Trigonometry

An angle is **in standard position** if its vertex is at the origin and its initial side is along the positive *x* axis. **Positive angles** are measured **counterclockwise** from the initial side. **Negative angles** are measured **clockwise**. We will typically use the Greek letter θ to denote an angle.



$\angle ABC$ is positive



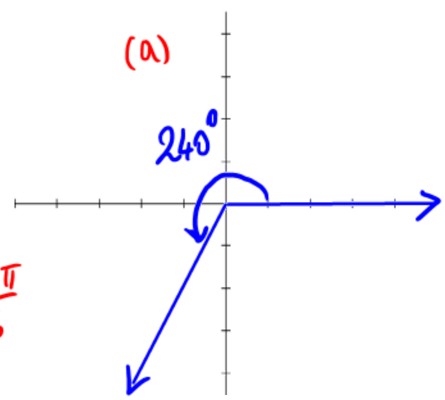


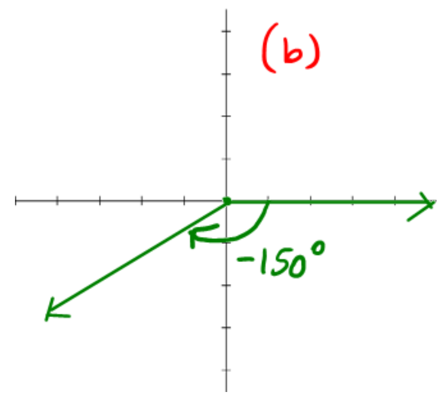
$\angle ABC$ is negative

Example 1: Sketch each angle in standard position.

a. $240° = 180° + 60°$

$240° = \pi + \frac{\pi}{3} = \frac{4\pi}{3}$



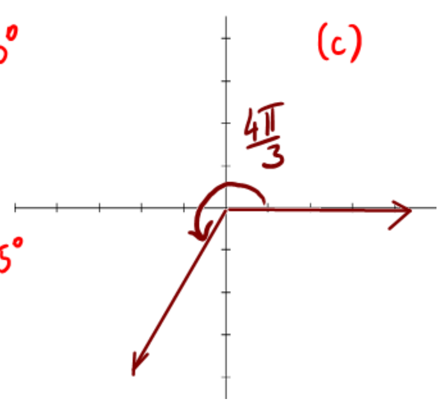


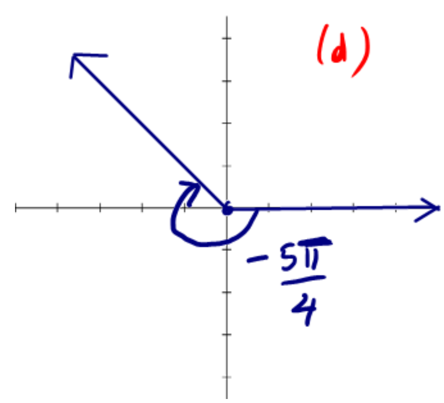
b. $-150° = -90° - 60°$

$-150° = -\frac{\pi}{2} - \frac{\pi}{3} = -\frac{5\pi}{6}$

c. $\frac{4\pi}{3} = \pi + \frac{\pi}{3}$

$\frac{4\pi}{3} = 180° + 60° = 240°$





d. $\frac{-5\pi}{4} = -\pi - \frac{\pi}{4}$

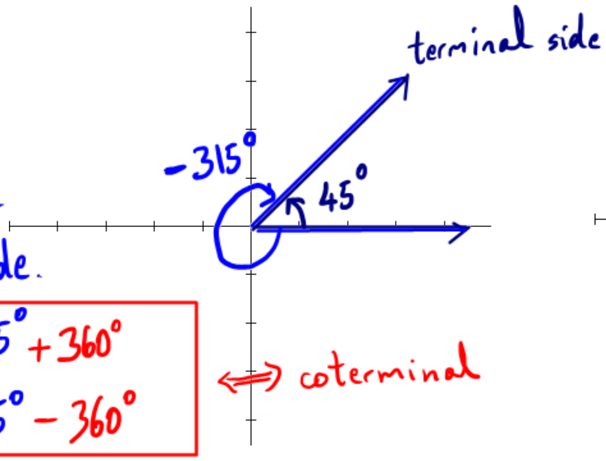
$-\frac{5\pi}{4} = -180° - 45° = -225°$

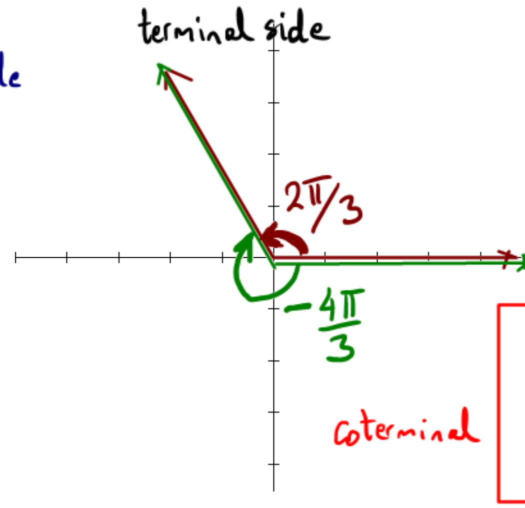
Angles that have the same terminal side are called **coterminal angles**. Measures of coterminal angles differ by a multiple of 360° if measured in degrees or by a multiple of 2π if measured in radians.

Demonstration

Both 45° and −315° have same terminal side.

$$\boxed{\begin{aligned} 45^\circ &= -315^\circ + 360^\circ \\ -315^\circ &= 45^\circ - 360^\circ \end{aligned}}$$





Both $\frac{2\pi}{3}$ & $-\frac{4\pi}{3}$ have same terminal side.

$$\boxed{\begin{aligned} \frac{2\pi}{3} &= -\frac{4\pi}{3} + 2\pi \\ -\frac{4\pi}{3} &= \frac{2\pi}{3} - 2\pi \end{aligned}}$$

Example 2: Find three angles, two positive and one negative that are coterminal with each angle.

a. 512° $= 360^\circ + 152^\circ$

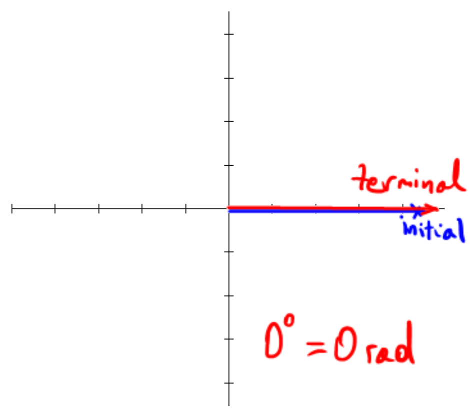
$512^\circ = \boxed{152^\circ} + \underbrace{360^\circ}$

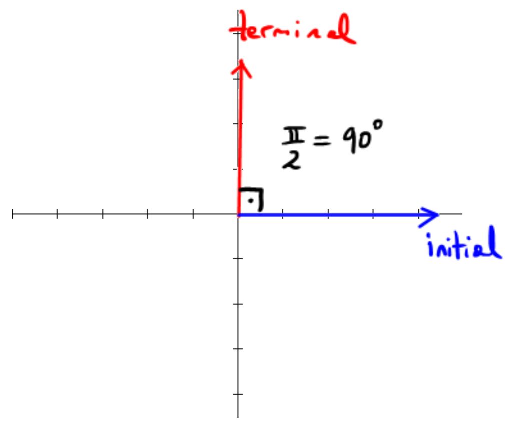
$$\left.\begin{aligned} \theta_1 &= 152^\circ = 512^\circ - 360^\circ \\ \theta_2 &= 512^\circ + 360^\circ = 872^\circ \\ \theta_3 &= 152^\circ - 360^\circ = -208^\circ \end{aligned}\right\} \text{coterminal}$$

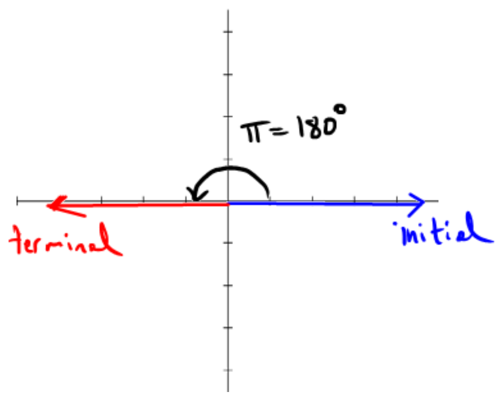
b. $\dfrac{-15\pi}{8}$

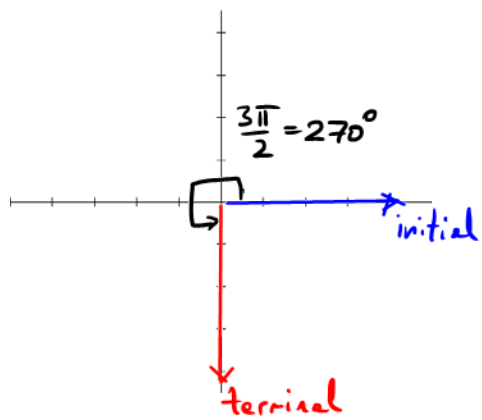
$$\left.\begin{aligned} \theta_1 &= -\frac{15\pi}{8} + 2\pi = -\frac{15\pi}{8} + \frac{16\pi}{8} = \frac{\pi}{8} \quad (+) \\ \theta_2 &= -\frac{15\pi}{8} + 4\pi = -\frac{15\pi}{8} + \frac{32\pi}{8} = \frac{17\pi}{8} \quad (+) \\ \theta_3 &= -\frac{15\pi}{8} - 2\pi = -\frac{15\pi}{8} - \frac{16\pi}{8} = -\frac{31\pi}{8} \quad (-) \end{aligned}\right\} \text{coterminal}$$

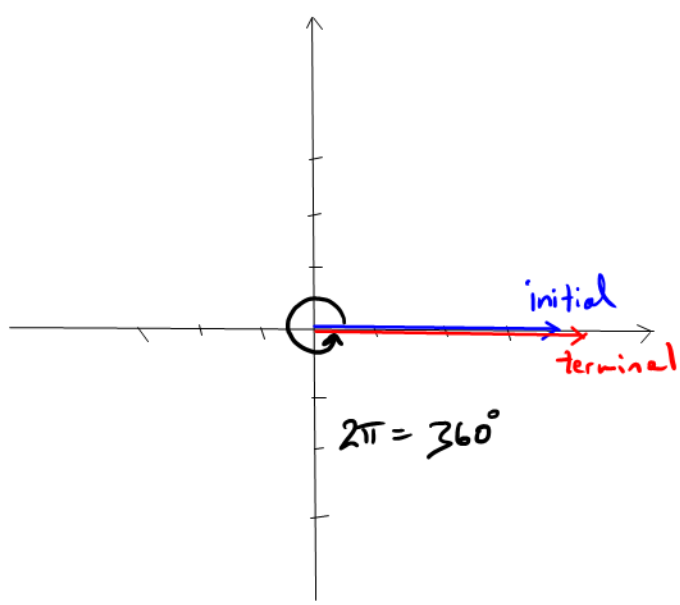
If an angle is in standard position and its terminal side lies along the x or y axis, then we call the angle a **quadrantal angle**. These are five basic quadrantal angles.



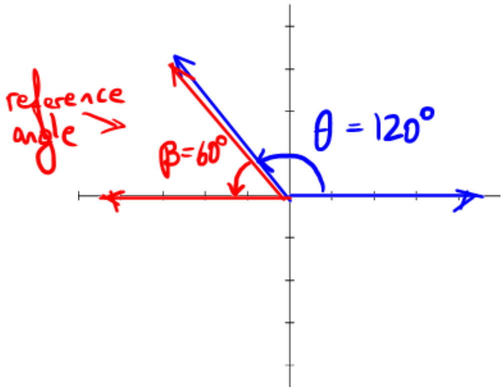


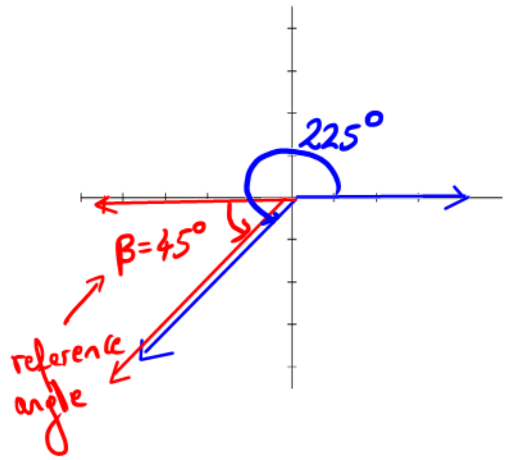






You will need to be able to work with reference angles. Suppose θ is an angle in standard position and θ is not a quadrantal angle. The **reference angle for** θ is the **acute angle of positive measure** that is formed by the terminal side of the angle and the *x* axis.



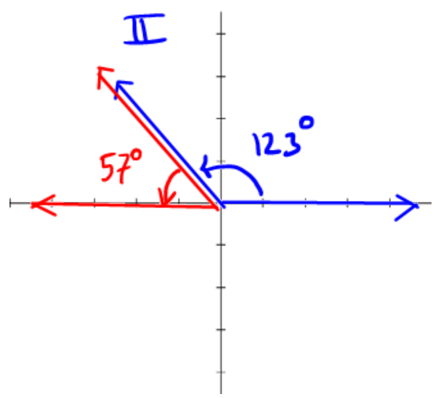


Example 3: Find the reference angle for each of these angles:

Quadrant II

a. 123°

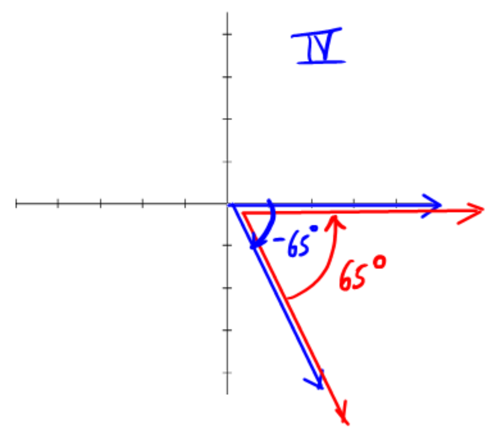
$180° - 123° = \boxed{57°}$



Quadrant IV

b. -65°

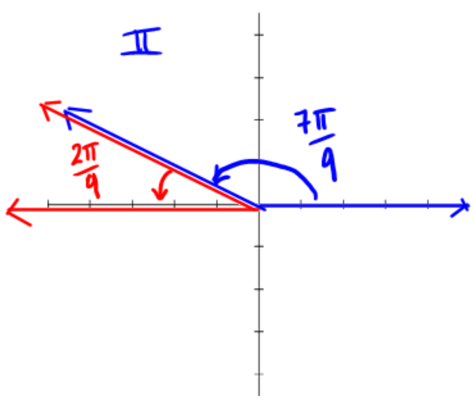
$-(-65°) = \boxed{65°}$



Quadrant II

c. $\frac{7\pi}{9}$

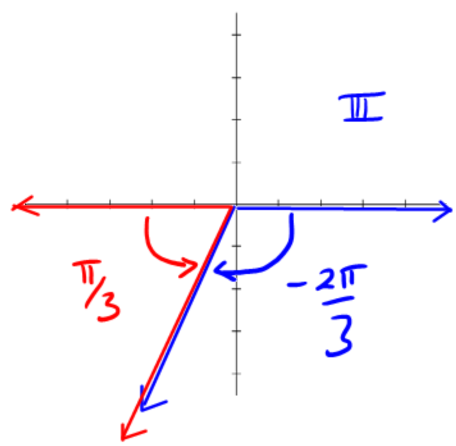
$\pi - \frac{7\pi}{9} = \boxed{\frac{2\pi}{9}}$



Quadrant III

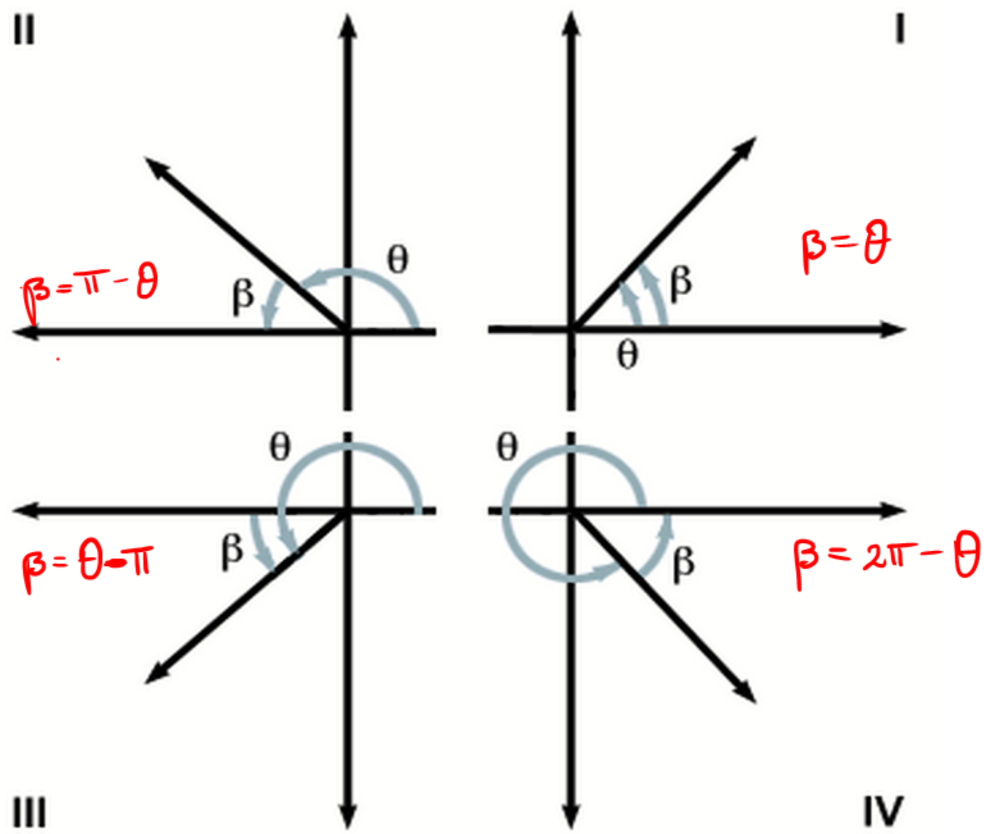
d. $\frac{-2\pi}{3}$

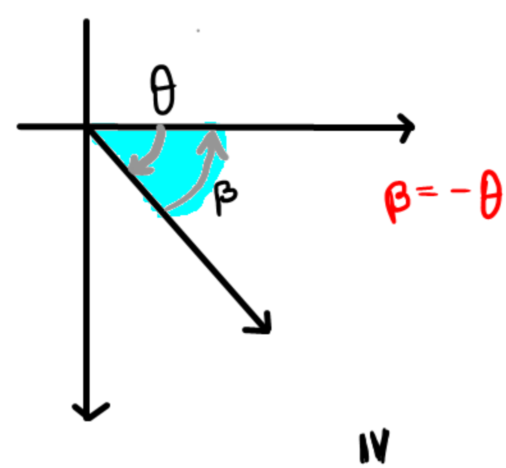
$\pi - \frac{2\pi}{3} = \boxed{\frac{\pi}{3}}$



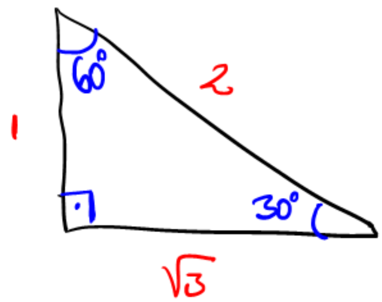
Reference angles:

Acute and positive





Popper # 13



You have to memorize the following:

C ① $\sin 60^\circ = \dfrac{Opp}{hyp} = \dfrac{\sqrt{3}}{2}$ — A. $\sqrt{3}$

B ② $\cos 60^\circ = \dfrac{Adj}{hyp} = \dfrac{1}{2}$ — B. $\dfrac{1}{2}$

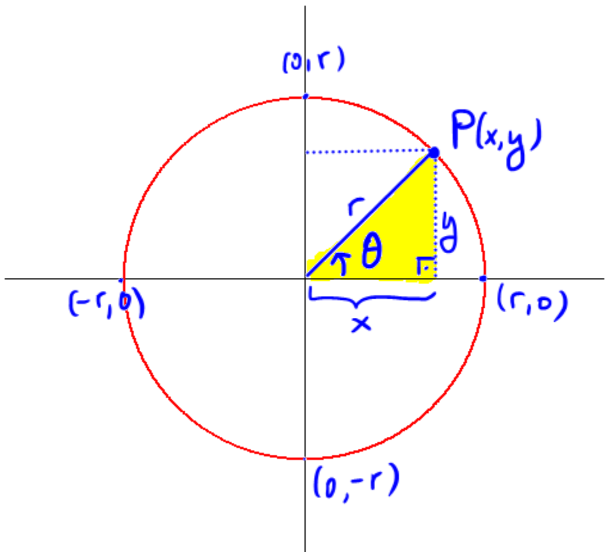
A ③ $\tan 60^\circ = \dfrac{Opp}{Adj} = \dfrac{\sqrt{3}}{1}$ — C. $\dfrac{\sqrt{3}}{2}$

A ④ $\cot 60^\circ = \dfrac{Adj}{Opp} = \dfrac{1}{\sqrt{3}} = \dfrac{\sqrt{3}}{3}$ — D. $\dfrac{2\sqrt{3}}{3}$

E ⑤ $\sec 60^\circ = \dfrac{hyp}{Adj} = \dfrac{2}{1}$ — E. 2

D ⑥ $\csc 60^\circ = \dfrac{hyp}{Opp} = \dfrac{2}{\sqrt{3}} = \dfrac{2\sqrt{3}}{3}$ — F. $\dfrac{\sqrt{3}}{3}$ $\longrightarrow$ call it A

We previously defined the six trigonometric functions of an angle as ratios of the lengths of the sides of a right triangle. Now we will look at them using a circle centered at the origin in the coordinate plane. This circle will have the equation $\boxed{x^2 + y^2 = r^2}$. If we select a point $P(x, y)$ on the circle and draw a ray from the origin through the point, we have created an angle in standard position. The length of the radius will be r.



The six trig functions of θ are defined as follows, using the circle above: Look at the yellow right triangle:

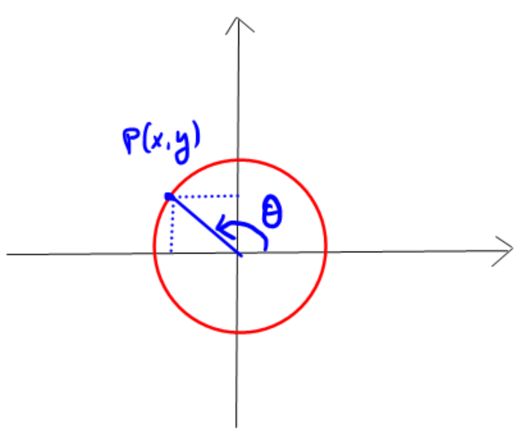
$$\sin\theta = \frac{y}{r} \qquad \csc\theta = \frac{r}{y},\ y \neq 0$$

$$\cos\theta = \frac{x}{r} \qquad \sec\theta = \frac{r}{x},\ x \neq 0$$

$$\tan\theta = \frac{y}{x},\ x \neq 0 \qquad \cot\theta = \frac{x}{y},\ y \neq 0$$

where $\boxed{r^2 = x^2 + y^2}$

If θ is a first quadrant angle, these definitions are consistent with the definitions given in Section 4.1.



What if θ is in 2^{nd} quadrant? Or 3^{rd} or 4^{th} ???

An **identity** is a statement that is true for all values of the variable. Here are some identities that follow from the definitions above.

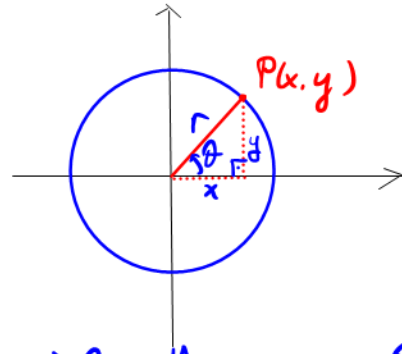
Memorize, they are very important

$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$

$$\cot\theta = \frac{\cos\theta}{\sin\theta} = \frac{1}{\tan\theta}$$

$$\csc\theta = \frac{1}{\sin\theta}$$

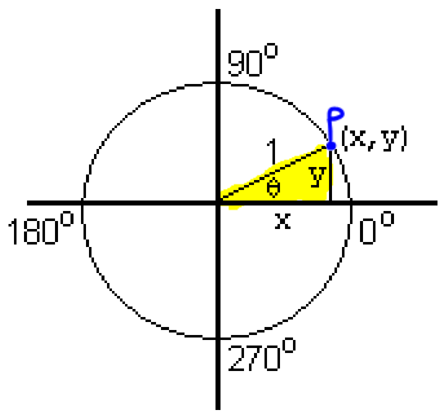
$$\sec\theta = \frac{1}{\cos\theta}$$



$x^2+y^2=r^2$

$\sin\theta = \frac{y}{r}$, $\cos\theta = \frac{x}{r}$

$\tan\theta = \frac{y}{x} = \frac{y/r}{x/r} = \frac{\sin\theta}{\cos\theta}$



$\Longleftrightarrow$ Every point on unit circle can be identified by trigonometric functions.

We will work most often with a **unit circle**, that is, a circle with radius 1. In this case, each value of *r* is 1. This adjusts the definitions of the trig functions as follows:

$$\sin\theta = y \qquad \csc\theta = \frac{1}{y},\ y \neq 0$$

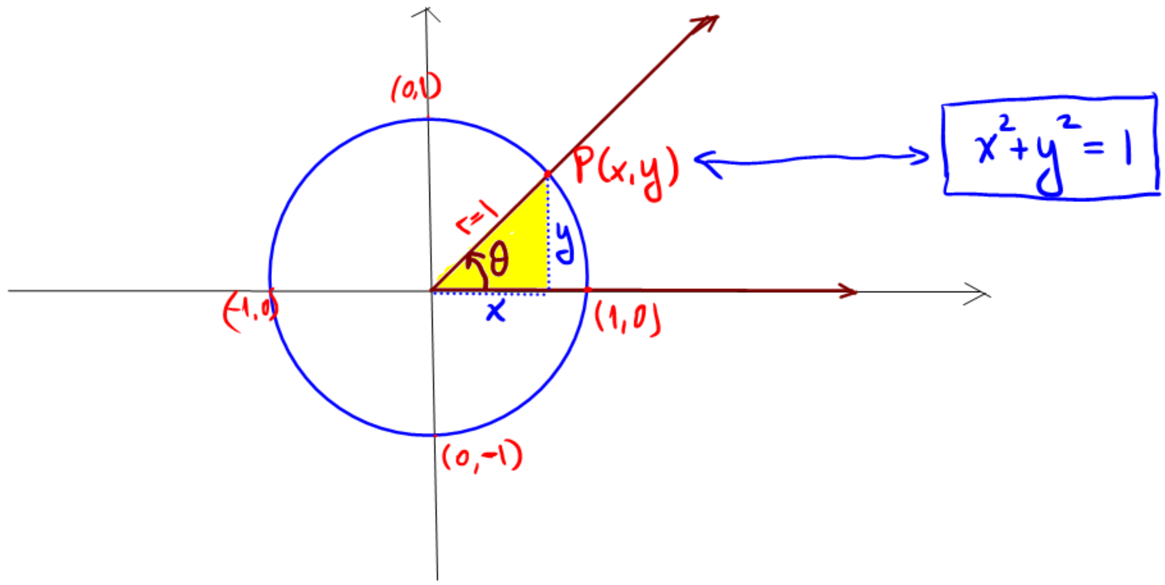
$$\cos\theta = x \qquad \sec\theta = \frac{1}{x},\ x \neq 0$$

$$\tan\theta = \frac{y}{x},\ x \neq 0 \qquad \cot\theta = \frac{x}{y},\ y \neq 0$$

Look at next page!

We'll get used to the unit circle. (center $(0,0)$, $r=1$) $x^2+y^2=1$

All trigonometric functions will be given by values of points on unit circle.



Trigonometric functions on unit circle:

$$\sin\theta = \frac{Opp}{Hyp.} = \frac{y}{r} = y$$

$$\cos\theta = \frac{Adj}{Hyp} = \frac{x}{r} = x$$

$$\tan\theta = \frac{Opp}{Adj} = \frac{y}{x}$$

$$\cot\theta = \frac{Adj}{Opp} = \frac{x}{y}$$

$$\sec\theta = \frac{Hyp}{Adj} = \frac{1}{x}$$

$$\csc\theta = \frac{Hyp}{Opp} = \frac{1}{y}$$

$x = \cos\theta$
$y = \sin\theta$ $\Longleftrightarrow$ every point (x,y) on the unit circle can be expressed as

$$P(\cos\theta, \sin\theta)$$

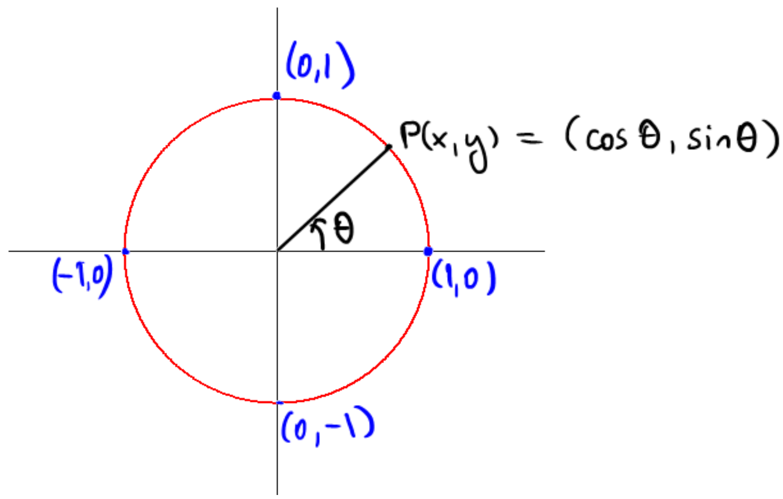
To be continued on Friday, 10/09

Trigonometric Functions of Quadrantal Angles and Special Angles

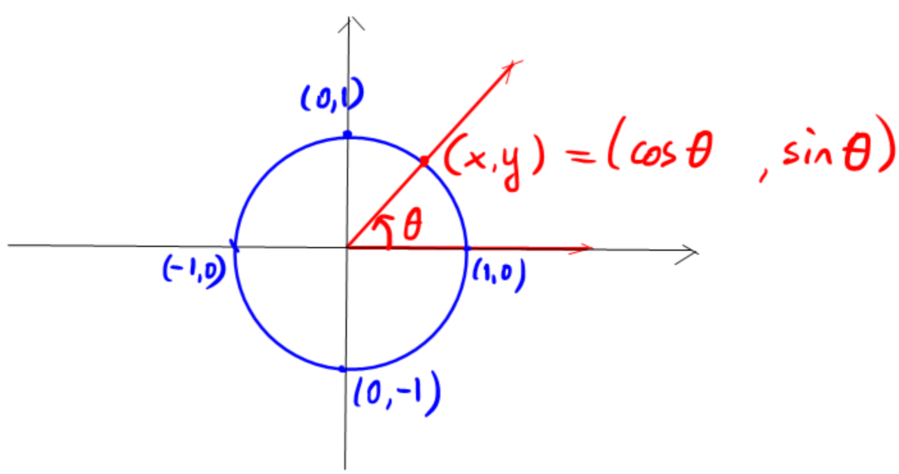
You will need to be able to find the trig functions of quadrantal angles and of angles measuring 30°, 45° or 60° without using a calculator.

Since $\sin\theta = y$ and $\cos\theta = x$, each ordered pair on the unit circle corresponds to $\underbrace{(\cos\theta, \sin\theta)}_{\text{don't forget}}$ of some angle θ.

We'll show the values for sine and cosine of the quadrantal angles on this graph. We'll also indicate where the trig functions are positive and where they are negative.

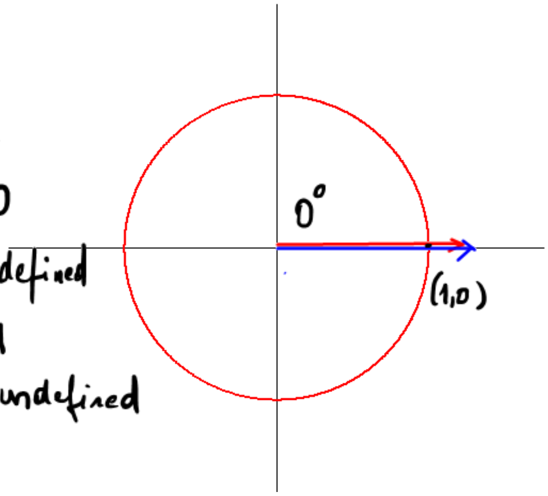


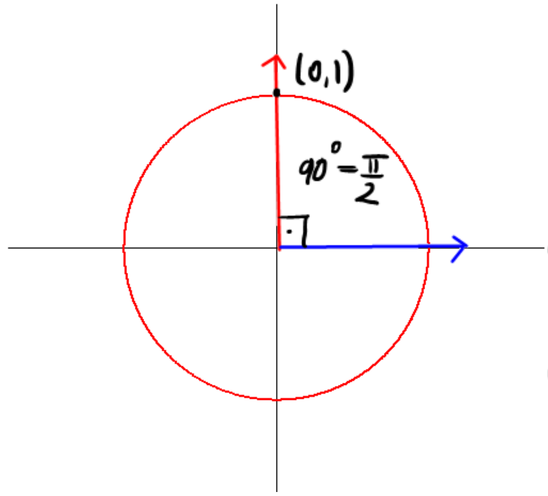
Get used:



Using the identities given above, you can find the other four trig functions of an angle, given just sine and cosine. Note that some values are not defined for quadrantal angles.

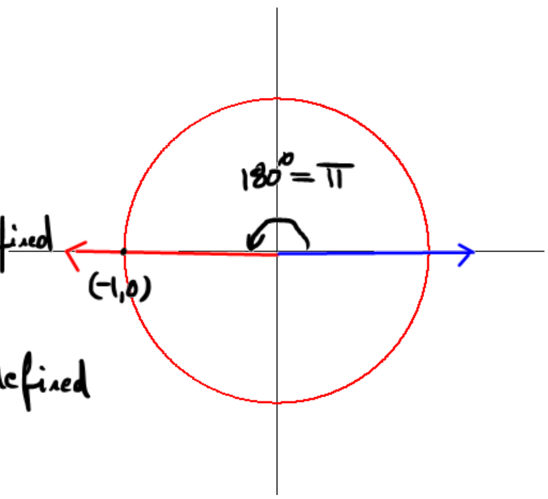
$\sin 0^\circ = 0$
$\cos 0^\circ = 1$
$\tan 0^\circ = 0$
$\cot 0^\circ = \text{undefined}$
$\sec 0^\circ = 1$
$\csc 0^\circ = \text{undefined}$

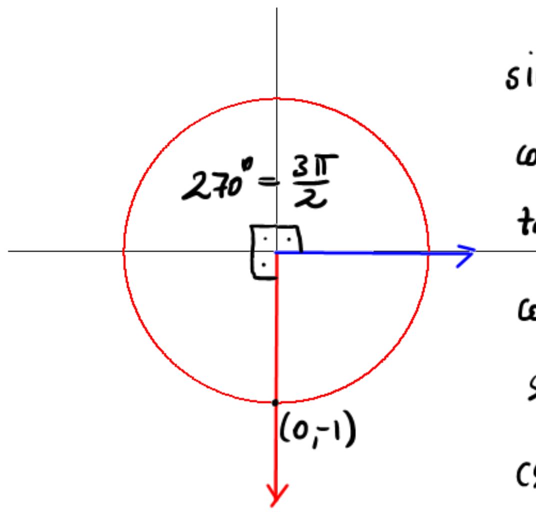




$\sin 90^\circ = 1$
$\cos 90^\circ = 0$
$\tan 90^\circ = \text{undefined}$
$\cot 90^\circ = 0$
$\sec 90^\circ = \text{undefined}$
$\csc 90^\circ = 1$

$\sin \pi = 0$
$\cos \pi = -1$
$\tan \pi = 0$
$\cot \pi = \text{undefined}$
$\sec \pi = -1$
$\csc \pi = \text{undefined}$





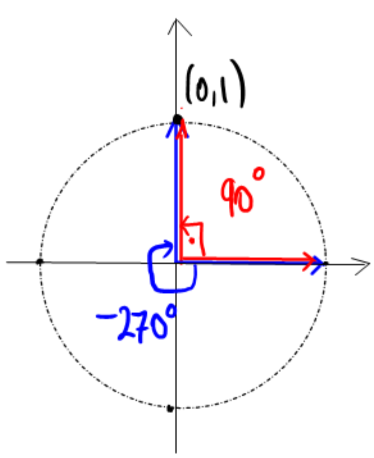
$\sin\left(\frac{3\pi}{2}\right) = -1$
$\cos\left(\frac{3\pi}{2}\right) = 0$
$\tan\left(\frac{3\pi}{2}\right) = \text{undefined}$
$\cot\left(\frac{3\pi}{2}\right) = 0$
$\sec\left(\frac{3\pi}{2}\right) = \text{undefined}$
$\csc\left(\frac{3\pi}{2}\right) = -1$

Note that the full angle $360^\circ = 2\pi$ matches the position of the 0° angle. Everything gets repeated.

Values of Trigonometric Functions for Quadrantal Angles

	0°	90°	180°	270°	360°
	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
Sine	0	1	0	-1	0
Cosine	1	0	-1	0	1
Tangent	0	undefined	0	undefined	0
Cotangent	undefined	0	undefined	0	undefined
Secant	1	undefined	-1	undefined	1
Cosecant	undefined	1	undefined	-1	undefined

Example 4: Sketch an angle measuring -270° in the coordinate plane. Then give the six trigonometric functions of the angle. Note that some of the functions may be undefined.



The trigonometric functions for the angle $\underline{\underline{-270^\circ}}$ are the same as those for angle $\underline{\underline{90^\circ}}$:

$\sin(-270^\circ) = 1$

$\cos(-270^\circ) = 0$

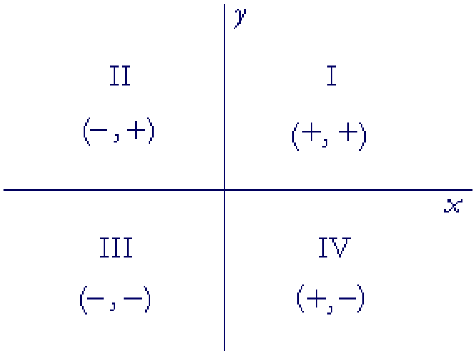
$\tan(-270^\circ) = \frac{\sin(-270^\circ)}{\cos(-270^\circ)} = \frac{1}{0}$ undefined

$\cot(-270^\circ) = \frac{\cos(-270^\circ)}{\sin(-270^\circ)} = \frac{0}{1} = 0$

$\sec(-270^\circ) =$ undefined

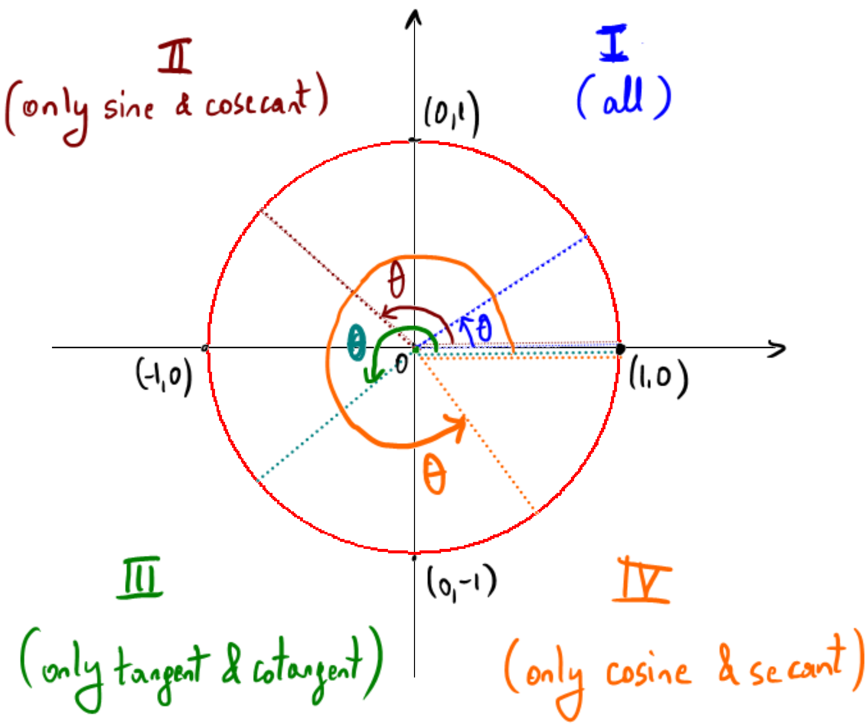
$\csc(-270^\circ) = 1$

Recall the signs of the points in each quadrant. Remember, that each point on the unit circle corresponds to an ordered pair, (cosine, sine).



$\sin\theta : +$
$\cos\theta : -$
$\tan\theta : -$
$\cot\theta : -$
$\sec\theta : -$
$\csc\theta : +$

$\sin\theta : +$
$\cos\theta : +$
$\tan\theta : +$
$\cot\theta : +$
$\sec\theta : +$
$\csc\theta : +$



$\sin\theta : -$
$\cos\theta : -$
$\tan\theta : +$
$\cot\theta : +$
$\sec\theta : -$
$\csc\theta : -$

$\sin\theta : -$
$\cos\theta : +$
$\tan\theta : -$
$\cot\theta : -$
$\sec\theta : +$
$\csc\theta : -$

Example 5: Name the quadrant in which both conditions are true:

a. $\cos\theta < 0$ and $\csc\theta > 0$.

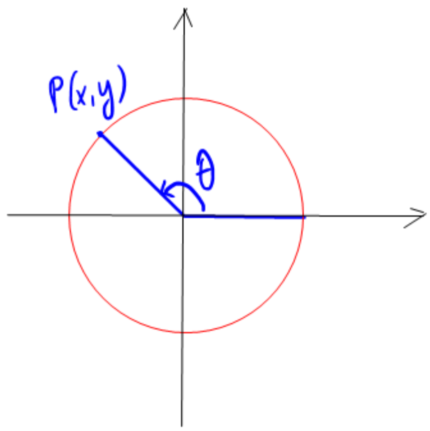
$\csc\theta = \frac{1}{\sin\theta} > 0 \iff \sin\theta > 0$ and $\cos\theta < 0 \iff \mathrm{II}$

b. $\sin\theta < 0$ and $\tan\theta < 0$

$\sin\theta < 0$ & $\tan\theta = \frac{\sin\theta}{\cos\theta} < 0 \Rightarrow \cos\theta > 0 \iff \mathrm{IV}$

This is a very typical type of problem you'll need to be able to work.

Example 6: Let $P(x, y)$ denote the point where the terminal side of an angle θ intersects the unit circle. If P is in quadrant II and $\boxed{y = \frac{5}{13}}$, find the six trig functions of angle θ.



$x = \cos\theta \leftarrow$ negative

$y = \sin\theta = \frac{5}{13} \leftarrow$ positive

$x^2 + y^2 = 1$

$x^2 + \left(\frac{5}{13}\right)^2 = 1 \iff x^2 = 1 - \frac{25}{169} = \frac{144}{169}$

$\Rightarrow x = -\sqrt{\frac{144}{169}} = -\frac{12}{13}$

$\Rightarrow \sin\theta = \frac{5}{13}$

$\cos\theta = -\frac{12}{13}$

$\tan\theta = \frac{5/13}{-12/13} = \frac{-5}{12}$

$\cot\theta = -\frac{12}{5}$

$\sec\theta = \frac{1}{-12/13} = -\frac{13}{12}$

$\csc\theta = \frac{1}{5/13} = \frac{13}{5}$

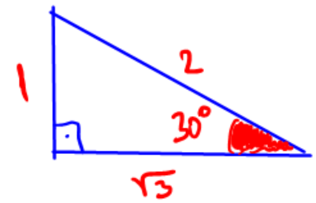
MEMORIZE (the logic)

You'll also need to be able to find the six trig functions of $30°, 60°$ and $45°$ angles. YOU MUST KNOW THESE!!!!!

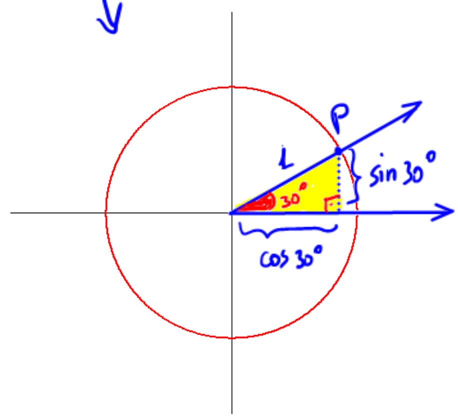
For a 30° angle:

unit circle

Recall 30°–60°–90° Δ:



$\sin 30° = \frac{1}{2}$, $\cos 30° = \frac{\sqrt{3}}{2}$



Point $P = (\cos 30°, \sin 30°)$
$= \left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$

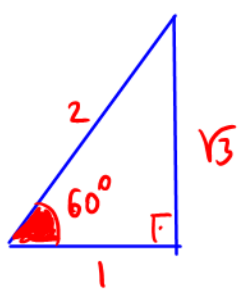
$$\boxed{\begin{aligned} \sin(30°) &= \frac{1}{2} \\ \cos(30°) &= \frac{\sqrt{3}}{2} \\ \tan(30°) &= \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3} \end{aligned}}$$

$$\csc(30°) = 2$$

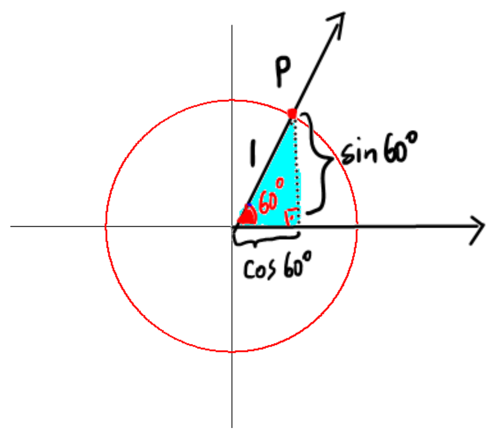
$$\sec(30°) = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$$

$$\cot(30°) = \sqrt{3}$$

For a 60° angle:



$\sin 60° = \frac{\sqrt{3}}{2}$, $\cos 60° = \frac{1}{2}$



$P = (\cos 60°, \sin 60°)$
$= \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$

$$\boxed{\begin{aligned} \sin(60°) &= \frac{\sqrt{3}}{2} \\ \cos(60°) &= \frac{1}{2} \\ \tan(60°) &= \sqrt{3} \end{aligned}}$$

$$\csc(60°) = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$$

$$\sec(60°) = 2$$

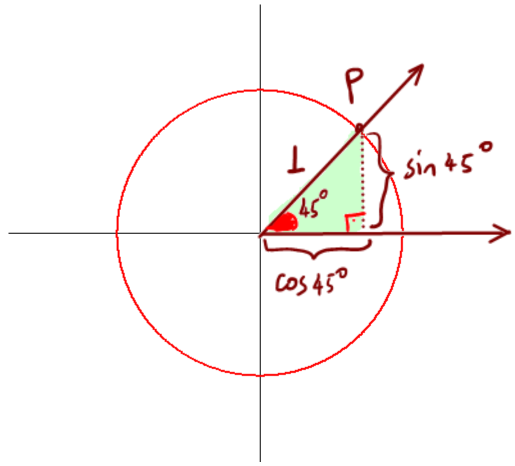
$$\cot(60°) = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$$

MEMORIZE (the logic)

For a 45° angle:

Recall 45°-45°-90° Δ:

$\sin 45° = \cos 45° = \frac{\sqrt{2}}{2}$



$P = (\cos 45°, \sin 45°)$

$= \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$

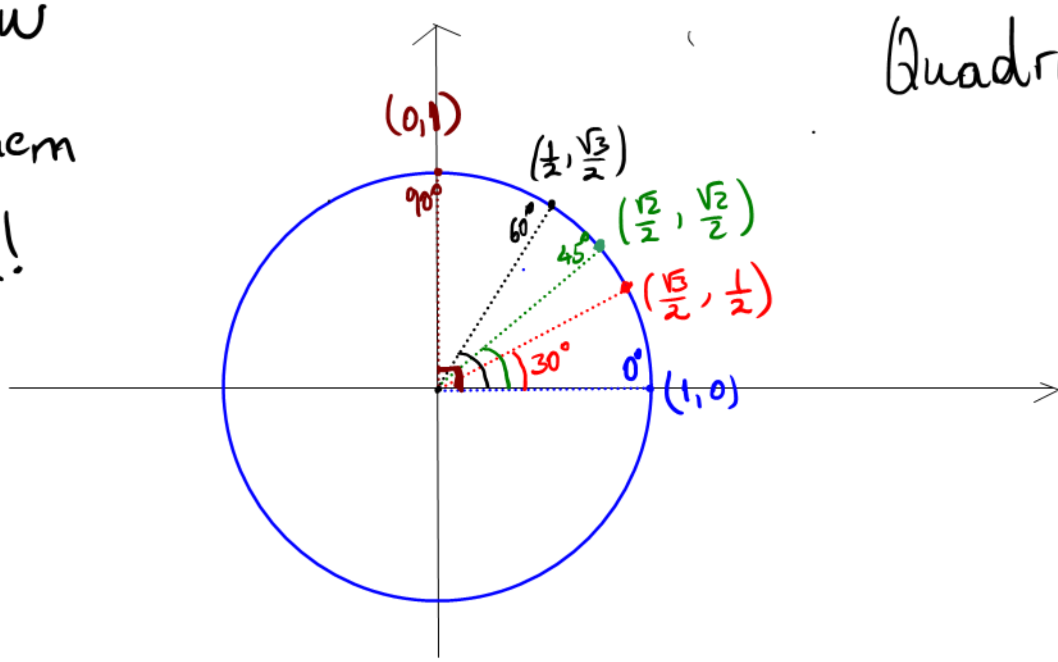
$$\sin(45°) = \frac{\sqrt{2}}{2} \qquad \csc(45°) = \sqrt{2}$$

$$\cos(45°) = \frac{\sqrt{2}}{2} \qquad \sec(45°) = \sqrt{2}$$

$$\tan(45°) = 1 \qquad \cot(45°) = 1$$

We know all of them now!

Quadrant I



Popper # 14

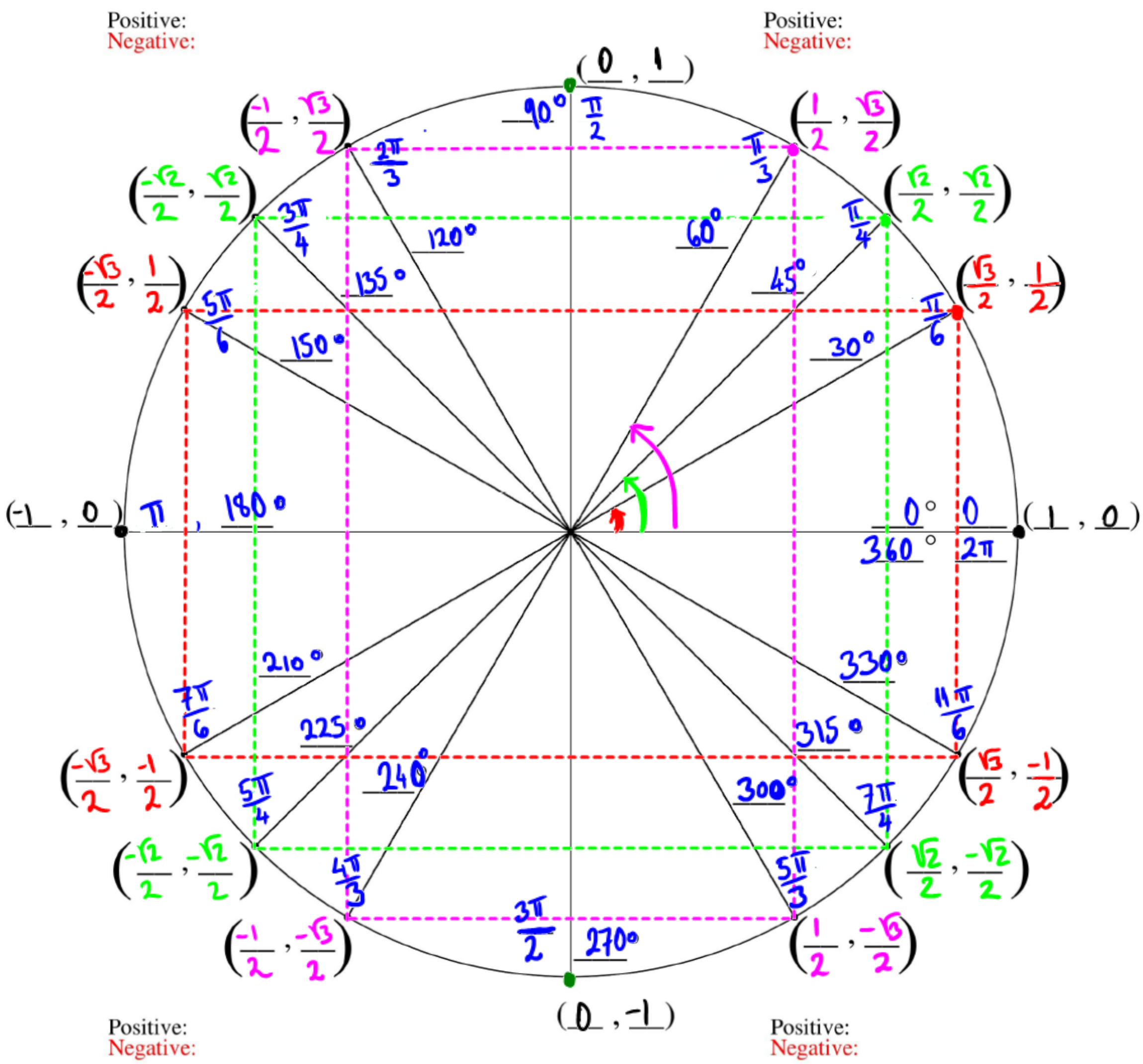
(free popper)

1. Bubble $\boxed{A}$
2. Bubble $\boxed{B}$.

Learn how to complete the unit circle.

How do we find the trigonometric functions of other special angles?

Method 1: Fill them in. Learn the patterns.



If I ask "find cos(225°)", the steps to follow are:

I. Locate the angle 225° (quadrant III)

II. Find its reference angle. (45°)

III. Give the answer using symmetry relationship of angle with its reference. (origin symmetry)

$\cos 225° = -\frac{\sqrt{2}}{2}$

Method 2: The Chart

Write down the angle measures, starting with 0° and continue until you reach 90°. Under these, write down the equivalent radian measures. Under these, write down the numbers from 0 to 4. Next, take the square root of the values and simplify if possible. Divide each value by 2. This gives you the sine value of each of the angles you need. To find the cosine values, write the previous line in the reverse order.

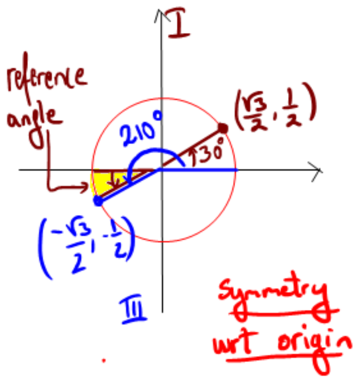
Now you have the sine and cosine values for the quadrantal angles and the special angles. From these, you can find the rest of the trig values for these angles. Write the problem in terms of the reference angle. Then use the chart you created to find the appropriate value.

Angles of Quadrant I

	0^0	30°	45°	60°	90^0
	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
Sine	**0**	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	**1**
Cosine	**1**	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	**0**
Tangent	**0**	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	**undefined**
Cotangent	**undefined**	$\sqrt{3}$	1	$\frac{\sqrt{3}}{3}$	**0**
Secant	**1**	$\frac{2\sqrt{3}}{3}$	$\sqrt{2}$	2	undefined
Cosecant	undefined	2	$\sqrt{2}$	$\frac{2\sqrt{3}}{3}$	**1**

The rest is symmetry of these values.

Example 7: Sketch an angle measuring 210° in the coordinate plane. Give the coordinates of the point where the terminal side of the angle intersects the unit circle. Then state the six trigonometric functions of the angle.



$\sin(210°) = -\sin(30°) = -\frac{1}{2}$

$\cos(210°) = -\cos(30°) = -\frac{\sqrt{3}}{2}$

$\tan(210°) = \frac{\sin(210°)}{\cos(210°)} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$

$\cot(210°) = \frac{1}{\tan(210°)} = \sqrt{3}$

$\csc(210°) = \frac{1}{\sin(210°)} = -2$

$\sec(210°) = \frac{1}{\cos(210°)} = -\frac{2\sqrt{3}}{3}$

Evaluating Trigonometric Functions Using Reference Angles

1. Determine the reference angle associated with the given angle.
2. Evaluate the given trigonometric function of the reference angle.
3. Affix the appropriate sign determined by the quadrant of the terminal side of the angle in standard position.

exercise on your own.

Example 8: Evaluate each.

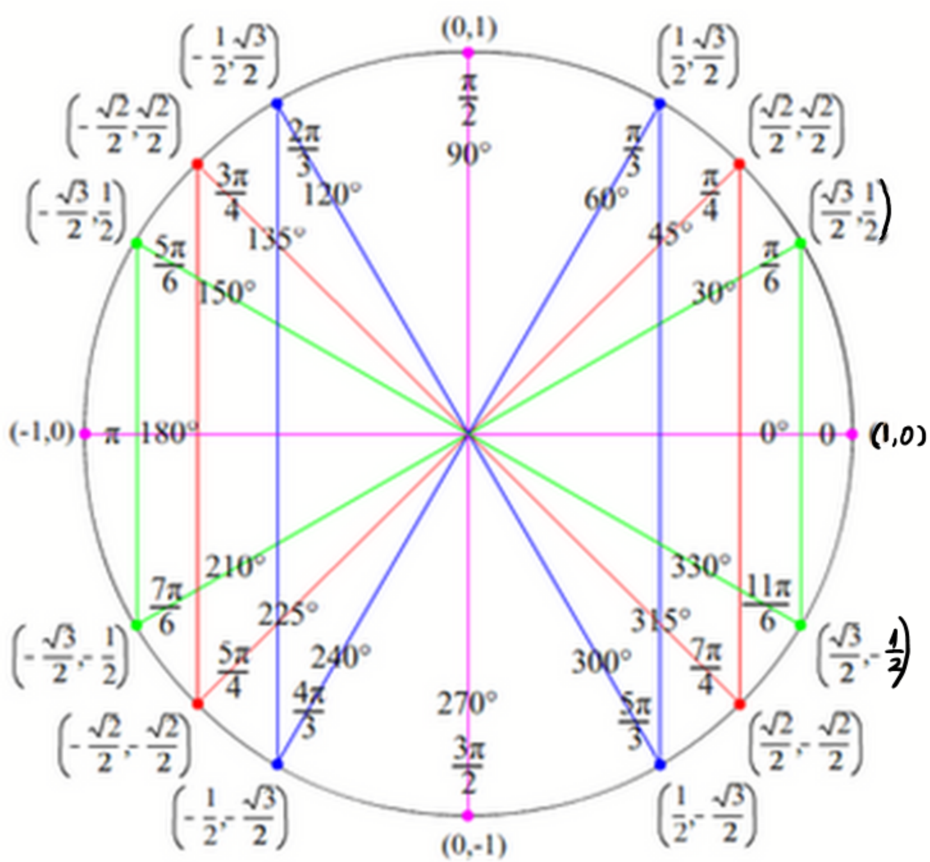
a. $\sin(300°) = -\sin(60°) = -\frac{1}{2}$

b. $\tan\left(\frac{3\pi}{4}\right) = \frac{\sin\left(\frac{3\pi}{4}\right)}{\cos\left(\frac{3\pi}{4}\right)} = -1$

c. $\sec(150°) = \frac{1}{\cos(150°)} = -\frac{2\sqrt{3}}{3}$

d. $\csc\left(\frac{-2\pi}{3}\right) = \frac{1}{\sin\left(\frac{-2\pi}{3}\right)} = \frac{-2\sqrt{3}}{3}$

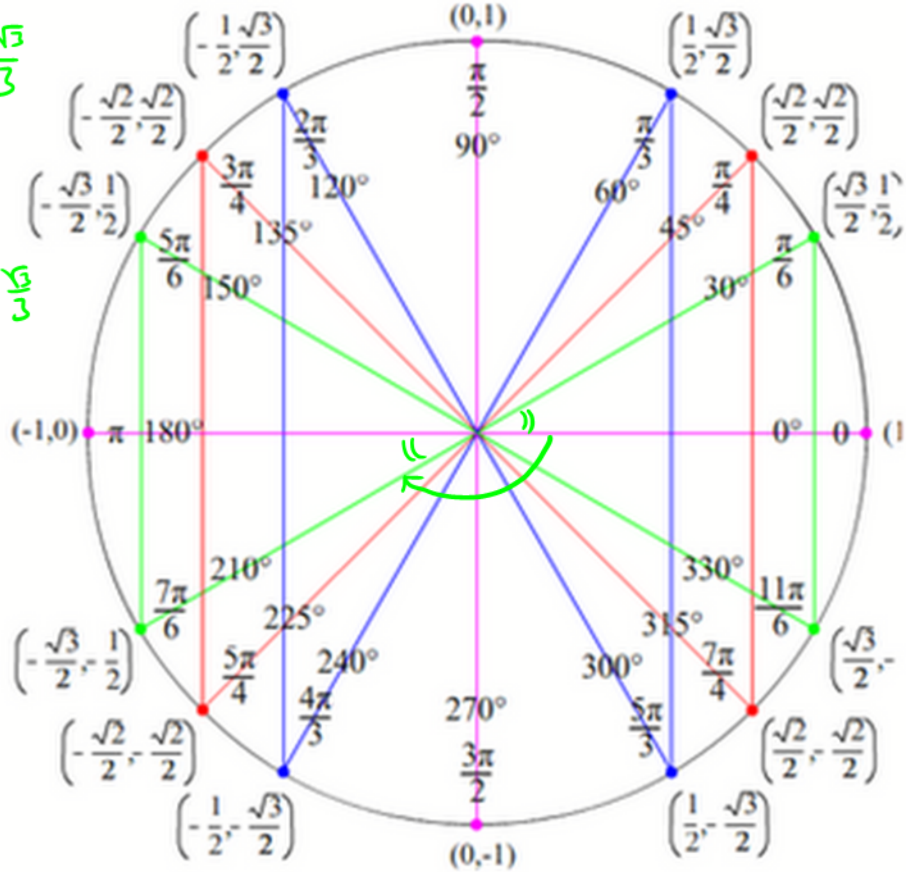
e. $\sec\left(\frac{11\pi}{6}\right) = \frac{1}{\cos\left(\frac{11\pi}{6}\right)} = \frac{2\sqrt{3}}{3}$



f. $\tan\left(\frac{7\pi}{6}\right) = \frac{\sin\left(\frac{7\pi}{6}\right)}{\cos\left(\frac{7\pi}{6}\right)} = \frac{-\frac{1}{2}}{-\frac{\sqrt{3}}{2}} = \frac{\sqrt{3}}{3}$

g. $\tan\left(-\frac{5\pi}{6}\right) = \frac{\sin\left(-\frac{5\pi}{6}\right)}{\cos\left(-\frac{5\pi}{6}\right)} = \frac{-\frac{1}{2}}{-\frac{\sqrt{3}}{2}} = \frac{\sqrt{3}}{3}$

h. $\tan\left(240^0\right) = \frac{\sin(240^\circ)}{\cos(240^\circ)} = \sqrt{3}$



i. $\cos\left(-150^0\right) = -\cos(30^\circ)$

$= -\frac{\sqrt{3}}{2}$

The reference angle is 30°.

Symmetry is origin.

$\Rightarrow \cos(-150^\circ) = -\cos(30^\circ) = -\frac{\sqrt{3}}{2}$.

$\sin(-150^\circ) = -\sin(30^\circ) = -\frac{1}{2}$

You do the same for any angle:

Locate, find reference, use symmetry and answer.

UNIT CIRCLE: STUDY, LEARN, APPLY!!!