PART II. SEQUENCES OF REAL NUMBERS

II.1. CONVERGENCE

Definition 1. *A* **sequence** *is a real-valued function* f *whose domain is the set positive integers* ($\mathbb{N}$). *The numbers* $f(1)$, $f(2)$, $\cdots$ *are called the* **terms** *of the sequence.*

Notation Function notation vs subscript notation:

$$f(1) \equiv s_1,\ f(2) \equiv s_2, \cdots,\ f(n) \equiv s_n,\ \cdots .$$

In discussing sequences the subscript notation is much more common than functional notation. We'll use subscript notation throughout our treatment of analysis.

Specifying a sequence There are several ways to specify a sequence.

1. By giving the function. For example:

   (a) $s_n = \dfrac{1}{n}$ or $\{s_n\} = \left\{\dfrac{1}{n}\right\}$. This is the sequence $\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots, \frac{1}{n}, \ldots\}$.

   (b) $s_n = \dfrac{n-1}{n}$. This is the sequence $\{0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots, \frac{n-1}{n}, \ldots\}$.

   (c) $s_n = (-1)^n n^2$. This is the sequence $\{-1, 4, -9, 16, \ldots, (-1)^n n^2, \ldots\}$.

2. By giving the first few terms to establish a pattern, leaving it to you to find the function. This is risky – it might not be easy to recognize the pattern and/or you can be misled.

   (a) $\{s_n\} = \{0, 1, 0, 1, 0, 1, \ldots\}$. The pattern here is obvious; can you devise the function? It's $s_n = \dfrac{1-(-1)^n)}{2}$ or $s_n = \begin{cases} 0, & n \text{ odd} \\ 1, & n \text{ even} \end{cases}$

   (b) $\{s_n\} = \left\{2, \frac{5}{2}, \frac{10}{3}, \frac{17}{4}, \frac{26}{5}, \ldots\right\}$, $\quad s_n = \dfrac{n^2+1}{n}$.

   (c) $\{s_n\} = \{2, 4, 8, 16, 32, \ldots\}$. What is s_6? What is the function? While you might say 64 and $s_n = 2^n$, the function I have in mind gives $s_6 = \pi/6$:

   $$s_n = 2^n + (n-1)(n-2)(n-3)(n-4)(n-5)\left[\frac{\pi}{720} - \frac{64}{120}\right]$$

3. By a recursion formula. For example:

   (a) $s_{n+1} = \dfrac{1}{n+1}\, s_n$, $\quad s_1 = 1$. The first 5 terms are $\left\{1, \frac{1}{2}, \frac{1}{6}, \frac{1}{24}, \frac{1}{120}, \ldots\right\}$. Assuming that the pattern continues $s_n = \dfrac{1}{n!}$.

   (b) $s_{n+1} = \frac{1}{2}(s_n + 1)$, $\quad s_1 = 1$. The first 5 terms are $\{1, 1, 1, 1, 1, \ldots\}$. Assuming that the pattern continues $s_n = 1$ for all n; $\{s_n\}$ is a "constant" sequence.

Definition 2. *A sequence* $\{s_n\}$ **converges** *to the number* s *if to each* $\epsilon > 0$ *there corresponds a positive integer* N *such that*

$$|s_n - s| < \epsilon \quad \textit{for all } n > N.$$

The number s *is called the* **limit** *of the sequence.*

Notation "$\{s_n\}$ converges to s" is denoted by

$$\lim_{n\to\infty} s_n = s, \quad \text{or by} \quad \lim s_n = s, \quad \text{or by} \quad s_n \to s.$$

A sequence that does not converge is said to **diverge**.

Examples Which of the sequences given above converge and which diverge; give the limits of the convergent sequences.

THEOREM 1. *If* $s_n \to s$ *and* $s_n \to t$, *then* $s = t$. *That is, the limit of a convergent sequence is unique.*

Proof: Suppose $s \neq t$. Assume $t > s$ and let $\epsilon = t - s$. Since $s_n \to s$, there exists a positive integer N_1 such that $|s - s_n| < \epsilon/2$ for all $n > N_1$. Since $s_n \to t$, there exists a positive integer N_2 such that $|t - s_n| < \epsilon/2$ for all $n > N_2$. Let $N = \max\{N_1, N_2\}$ and choose a positive integer $k > N$. Then

$$t - s = |t - s| = |t - s_k + s_k - s| \leq |t - s_k| + |s - s_k| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon = t - s,$$

a contradiction. Therefore, $s = t$.

THEOREM 2. *If* $\{s_n\}$ *converges, then* $\{s_n\}$ *is bounded.*

Proof: Suppose $s_n \to s$. There exists a positive integer N such that $|s - s_n| < 1$ for all $n > N$. Therefore, it follows that

$$|s_n| = |s_n - s + s| \leq |s_n - s| + |s| < 1 + |s| \quad \text{for all} \quad n > N.$$

Let $M = \max\{|s_1|,\ |s_2|,\ \ldots,\ |s_N|,\ 1 + |s|\}$. Then $|s_n| < M$ for all n. Therefore $\{s_n\}$ is bounded.

THEOREM 3. *Let* $\{s_n\}$ *and* $\{a_n\}$ *be sequences and suppose that there is a positive number* k *and a positive integer* N *such that*

$$|s_n| \leq k\, a_n \quad \textit{for all } n > N.$$

If $a_n \to 0$, *then* $s_n \to 0$.

Proof: Note first that $a_n \geq 0$ for all $n > N$. Since $a_n \to 0$, there exists a positive integer N_1 such that $|a_n| < \epsilon/k$. Without loss of generality, assume that $N_1 \geq N$. Then, for all $n > N_1$,

$$|s_n - 0| = |s_n| \leq k\, a_n < k\,\frac{\epsilon}{k} = \epsilon.$$

Therefore, $s_n \to 0$.

Corollary Let $\{s_n\}$ and $\{a_n\}$ be sequences and let $s \in \mathbb{R}$. Suppose that there is a positive number k and a positive integer N such that

$$|s_n - s| \le k\, a_n \quad \text{for all } n > N.$$

If $a_n \to 0$, then $s_n \to s$.

Exercises 2.1

1. True – False. Justify your answer by citing a theorem, giving a proof, or giving a counterexample.

   (a) If $s_n \to s$, then $s_{n+1} \to s$.

   (b) If $s_n \to s$ and $t_n \to s$, then there is a positive integer N such that $s_n = t_n$ for all $n > N$.

   (c) Every bounded sequence converges

   (d) If to each $\epsilon > 0$ there is a positive integer N such that $n > N$ implies $s_n < \epsilon$, then $s_n \to 0$.

   (e) If $s_n \to s$, then s is an accumulation point of the set $S = \{s_1,\ s_2,\ \cdots\}$.

2. Prove that $\lim \dfrac{3n+1}{n+2} = 3$.

3. Prove that $\lim \dfrac{\sin n}{n} = 0$.

4. Prove or give a counterexample:

   (a) If $\{s_n\}$ converges, then $\{|s_n|\}$ converges.

   (b) If $\{|s_n|\}$ converges, then $\{s_n\}$ converges.

5. Give an example of:

   (a) A convergent sequence of rational numbers having an irrational limit.

   (b) A convergent sequence of irrational numbers having a rational limit.

6. Give the first six terms of the sequence and then give the nth term

   (a) $s_1 = 1, \quad s_{n+1} = \frac{1}{2}(s_n + 1)$

   (b) $s_1 = 1, \quad s_{n+1} = \frac{1}{2}s_n + 1$

   (c) $s_1 = 1, \quad s_{n+1} = 2s_n + 1$

7. use induction to prove the following assertions:

   (a) If $s_1 = 1$ and $s_{n+1} = \dfrac{n+1}{2n}\, s_n$, then $s_n = \dfrac{n}{2^{n-1}}$.

   (b) If $s_1 = 1$ and $s_{n+1} = s_n - \dfrac{1}{n(n+1)}$, then $s_n = \dfrac{1}{n}$.

8. Let r be a real number, $r \neq 0$. Define a sequence $\{S_n\}$ by

$$\begin{aligned} S_1 &= 1 \\ S_2 &= 1 + r \\ S_3 &= 1 + r + r^2 \\ &\vdots \\ S_n &= 1 + r + r^2 + \cdots + r^{n-1} \\ &\vdots \end{aligned}$$

(a) Suppose $r = 1$. What is S_n for $S_n = 1, 2, 3, \ldots$?

(b) Suppose $r \neq 1$. Find a formula for S_n.

9. Set $a_n = \dfrac{1}{n(n+1)}$, $n = 1, 2, 3, \ldots$, and form the sequence

$$\begin{aligned} S_1 &= a_1 \\ S_2 &= a_1 + a_2 \\ S_3 &= a_1 + a_2 + a_3 \\ &\vdots \\ S_n &= a_1 + a_2 + a_3 + \cdots + a_n \\ &\vdots \end{aligned}$$

Find a formula for S_n.

II.2. LIMIT THEOREMS

THEOREM 4. *Suppose* $s_n \to s$ *and* $t_n \to t$. *Then:*

1. $s_n + t_n \to s + t$.

2. $s_n - t_n \to s - t$.

3. $s_n t_n \to st$.

Special case: $ks_n \to ks$ *for any number* k.

4. $s_n/t_n \to s/t$ *provided* $t \neq 0$ *and* $t_n \neq 0$ *for all* n.

THEOREM 5. *Suppose* $s_n \to s$ *and* $t_n \to t$. *If* $s_n \leq t_n$ *for all* n, *then* $s \leq t$.

Proof: Suppose $s > t$. Let $\epsilon = \dfrac{s-t}{2}$. Since $s_n \to s$, there exists a positive integer N_1 such that $|s_n - s| < \epsilon$ for all $n > N_1$. This implies that $s - \epsilon < s_n < s + \epsilon$ for all $n > N_1$. Similarly, there exists a positive integer N_2 such that $t - \epsilon < t_n < t + \epsilon$ for all $n > N_2$. Let $N = \max\{N_1,\ N_2\}$. Then, for all $n > N$, we have

$$t_n < t + \epsilon = t + \frac{s-t}{2} = \frac{s+t}{2} = s - \epsilon < s_n$$

which contradicts the assumption $s_n \leq t_n$ for all n.

Corollary Suppose $t_n \to t$. If $t_n \geq 0$ for all n, then $t \geq 0$.

Infinite Limits

Definition 3. *A sequence* $\{s_n\}$ **diverges to** $+\infty$ $(s_n \to +\infty)$ *if to each real number* M *there is a positive integer* N *such that* $s_n > M$ *for all* $n > N$. $\{s_n\}$ **diverges to** $-\infty$ $(s_n \to -\infty)$ *if to each real number* M *there is a positive integer* N *such that* $s_n < M$ *for all* $n > N$.

THEOREM 6. *Suppose that* $\{s_n\}$ *and* $\{t_n\}$ *are sequences such that* $s_n \leq t_n$ *for all* n.

1. *If* $s_n \to +\infty$, *then* $t_n \to +\infty$.
2. *If* $t_n \to -\infty$, *then* $s_n \to -\infty$.

THEOREM 7. *Let* $\{s_n\}$ *be a sequence of positive numbers. Then* $s_n \to +\infty$ *if and only if* $1/s_n \to 0$.

Proof: Suppose $s_n \to \infty$. Let $\epsilon > 0$ and set $M = 1/\epsilon$. Then there exists a positive integer N such that $s_n > M$ for all $n > N$. Since $s_n > 0$,

$$1/s_n < 1/M = \epsilon \quad \text{for all } n > N$$

which implies $1/s_n \to 0$.

Now suppose that $1/s_n \to 0$. Choose any positive number M and let $\epsilon = 1/M$. Then there exists a positive integer N such that

$$0 < \frac{1}{s_n} < \epsilon = \frac{1}{M} \quad \text{for all} \quad n > N. \quad \text{that is,} \quad \frac{1}{s_n} < \frac{1}{M}.$$

Since $s_n > 0$ for all n, $1/s_n < 1/M$ for all $n > N$ implies $s_n > M$ for all $n > N$. Therefore, $s_n \to \infty$.

Exercises 2.2

1. Prove or give a counterexample.
   (a) If $s_n \to s$ and $s_n > 0$ for all n, then $s > 0$.
   (b) If $\{s_n\}$ and $\{t_n\}$ are divergent sequences, then $\{s_n + t_n\}$ is divergent.
   (c) If $\{s_n\}$ and $\{t_n\}$ are divergent sequences, then $\{s_n t_n\}$ is divergent.
   (d) If $\{s_n\}$ and $\{s_n + t_n\}$ are convergent sequences, then $\{t_n\}$ is convergent.
   (e) If $\{s_n\}$ and $\{s_n t_n\}$ are convergent sequences, then $\{t_n\}$ is convergent.
   (f) If $\{s_n\}$ is not bounded above, then $\{s_n\}$ diverges to $+\infty$.
2. Determine the convergence or divergence of $\{s_n\}$. Find any limits that exist.

(a) $s_n = \dfrac{3-2n}{1+n}$ (b) $s_n = \dfrac{(-1)^n}{n+2}$

(c) $s_n = \dfrac{(-1)^n n}{2n-1}$ (d) $s_n = \dfrac{2^{3n}}{3^{2n}}$

(e) $s_n = \dfrac{n^2-2}{n+1}$ (f) $s_n = \dfrac{1+n+n^2}{1+3n}$

3. Prove the following:

   (a) $\lim_{n\to\infty} \left(\sqrt{n^2+1} - n\right) = 0.$

   (b) $\lim_{n\to\infty} \left(\sqrt{n^2+n} - n\right) = \dfrac{1}{2}.$

4. Prove Theorem 4.

5. Prove Theorem 6.

6. Let $\{s_n\}$, $\{t_n\}$, and $\{u_n\}$ be sequnces such that $s_n \le t_n \le u_n$ for all n. Prove that if $s_n \to L$ and $u_n \to L$, then $t_n \to L$.

II.3. MONOTONE SEQUENCES AND CAUCHY SEQUENCES

Monotone Sequences

Definition 4. *A sequence $\{s_n\}$ is* **increasing** *if $s_n \le s_{n+1}$ for all n; $\{s_n\}$ is* **decreasing** *if $s_n \ge s_{n+1}$ for all n. A sequence is* **monotone** *if it is increasing or if it is decreasing.*

Examples

(a) $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots, \frac{1}{n}, \ldots$ is a decreasing sequence.

(b) $2, 4, 8, 16, \ldots, 2^n, \ldots$ is an increasing sequence.

(c) $1, 1, 3, 3, 5, 5, \ldots, 2n-1, 2n-1, \ldots$ is an increasing sequence.

(d) $1, \frac{1}{2}, 3, \frac{1}{4}, 5, \ldots$ is not monotonic.

Some methods for showing monotonicity:

(a) To show that a sequence is increasing, show that $\dfrac{s_{n+1}}{s_n} \ge 1$ for all n. For decreasing, show $\dfrac{s_{n+1}}{s_n} \le 1$ for all n.

The sequence $s_n = \dfrac{n}{n+1}$ is increasing: Since

$$\frac{s_{n+1}}{s_n} = \frac{(n+1)/(n+2)}{n/(n+1)} = \frac{n+1}{n+2} \cdot \frac{n+1}{n} = \frac{n^2+2n+1}{n^2+2n} > 1$$

(b) By induction. For example, let $\{s_n\}$ be the sequence defined recursively by

$$s_{n+1} = 1 + \sqrt{s_n}, \quad s_1 = 1.$$

We show that $\{s_n\}$ is increasing. Let S be the set of positive integers for which $s_{k+1} \geq s_k$. Since $s_2 = 1 + \sqrt{1} = 2 > 1$, $1 \in S$. Assume that $k \in S$; that is, that $s_{k+1} \geq s_k$. Consider s_{k+2}:

$$s_{k+2} = 1 + \sqrt{s_{k+1}} \geq 1 + \sqrt{s_k} = s_{k+1}.$$

Therefore, $s_{k+1} \in S$ and $\{s_n\}$ is increasing.

THEOREM 8. *A monotone sequence is convergent if and only if it is bounded.*

Proof: Let $\{s_n\}$ be a monotone sequence.

If $\{s_n\}$ is convergent, then it is bounded (Theorem 2).

Now suppose that $\{s_n\}$ is a bounded, monotone sequence. In particular, suppose $\{s_n\}$ is increasing. Let $u = \sup\{s_n\}$ and let ϵ be a positive number. Then there exists a positive integer N such that $u - \epsilon < s_N \leq u$. Since $\{s_n\}$ is increasing, $u - \epsilon < s_n \leq u$ for all $n > N$. Therefore, $|u - s_n| < \epsilon$ for all $n > N$ and $s_n \to u$.

A similar argument holds for the case $\{s_n\}$ decreasing.

THEOREM 9. *(a) If $\{s_n\}$ is increasing and unbounded, then $s_n \to +\infty$.*

(b) If $\{s_n\}$ is decreasing and unbounded, then $s_n \to -\infty$.

Proof: (a) Since $\{s_n\}$ is increasing, $s_n \geq s_1$ for all n. Therefore, $\{s_n\}$ is bounded below. Since $\{s_n\}$ is unbounded, it is unbounded above and to each positive number M there is a positive integer N such that $s_N > M$. Again, since $\{s_n\}$ is increasing, $s_n \geq s_N > M$ for all $n > N$. Therefore $s_n \to \infty$.

The proof of (b) is left as an exercise.

Cauchy Sequences

Definition 5. *A sequence $\{s_n\}$ is a* **Cauchy sequence** *if to each $\epsilon > 0$ there is a positive integer N such that*

$$m,\ n > N \quad \textit{implies} \quad |s_n - s_m| < \epsilon.$$

THEOREM 10. *Every convergent sequence is a Cauchy sequence.*

Proof: Suppose $s_n \to s$. Let $\epsilon > 0$. There exists a positive integer N such that $|s - s_n| < \epsilon/2$ for all $n > N$. Let $n,\ m > N$. Then

$$|s_m - s_n| = |s_m - s + s - s_n| \leq |s_m - s| + |s - s_n| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Therefore $\{s_n\}$ is a Cauchy sequence.

THEOREM 11. *Every Cauchy sequence is bounded.*

Proof: Let $\{s_n\}$ be a Cauchy sequence. There exists a positive integer N such that $|s_n - s_m| < 1$ whenever $n,\ m > M$. Therefore

$$|s_n| = |s_n - s_{N+1} + s_{N+1}| \leq |s_n - s_{N+1}| + |s_{N+1}| < 1 + |s_{N+1}| \quad \text{for all} \quad n > N.$$

Now let $M = \max\{|s_1|,\ |s_2|,\ \ldots,\ |s_N|,\ 1 + |s_{N+1}|\}$. Then $|s_n| \leq M$ for all n.

THEOREM 12. *A sequence $\{s_n\}$ is convergent if and only if it is a Cauchy sequence.*

Exercises 2.3

1. True – False. Justify your answer by citing a theorem, giving a proof, or giving a counter-example.
   (a) If a monotone sequence is bounded, then it is convergent.
   (b) If a bounded sequence is monotone, then it is convergent.
   (c) If a convergent sequence is monotone, then it is bounded.
   (d) If a convergent sequence is bounded, then it is monotone.
2. Give an example of a sequence having the given properties.
   (a) Cauchy, but not monotone.
   (b) Monotone, but not Cauchy.
   (c) Bounded, but not Cauchy.
3. Show that the sequence $\{s_n\}$ defined by $s_1 = 1$ and $s_{n+1} = \frac{1}{4}(s_n + 5)$ is monotone and bounded. Find the limit.
4. Show that the sequence $\{s_n\}$ defined by $s_1 = 2$ and $s_{n+1} = \sqrt{2s_n + 1}$ is monotone and bounded. Find the limit.
5. Show that the sequence $\{s_n\}$ defined by $s_1 = 1$ and $s_{n+1} = \sqrt{s_n + 6}$ is monotone and bounded. Find the limit.
6. Prove that a bounded decreasing sequence converges to its greatest lower bound.
7. Prove Theorem 9 (b).

II.4. SUBSEQUENCES

Definition 6. *Given a sequence $\{s_n\}$. Let $\{n_k\}$ be a sequence of positive integers such that $n_1 < n_2 < n_3 < \cdots$. The sequence $\{s_{n_k}\}$ is called a* **subsequence** *of $\{s_n\}$.*

Examples

THEOREM 13. *If $\{s_n\}$ converges to s, then every subsequence $\{s_{n_k}$ of $\{s_n\}$ also converges to s.*

Corollary If $\{s_n\}$ has a subsequence $\{t_n\}$ that converges to α and a subsequence $\{u_n\}$ that converges to β with $\alpha \neq \beta$, then $\{s_n\}$ does not converge.

THEOREM 14. *Every bounded sequence has a convergent subsequence.*

THEOREM 15. *Every unbounded sequence has a monotone subsequence that diverges either to* $+\infty$ *or to* $-\infty$*.*

Limit Superior and Limit Inferior

Definition 7. *Let* $\{s_n\}$ *be a bounded sequence. A number* α *is a* **subsequential limit** *of* $\{s_n\}$ *if there is a subsequence* $\{s_{n_k}\}$ *of* $\{s_n\}$ *such that* $s_{n_k} \to \alpha$*.*

Examples

Let $\{s_n\}$ be a bounded sequence. Let

$$S = \{\alpha : \alpha \text{ is a subsequential limit of } \{s_n\}.$$

Then:

1. $S \neq \emptyset$.
2. S is a bounded set.

Definition 8. *Let* $\{s_n\}$ *be a bounded sequence and let* S *be its set of subsequential limits. The* **limit superior** *of* $\{s_n\}$ *(denoted by lim sup* s_n*) is*

$$\textit{lim sup } s_n = \textit{sup } S.$$

The **limit inferior** *of* $\{s_n\}$ *(denoted by lim inf* s_n*) is*

$$\textit{lim inf } s_n = \textit{inf } S.$$

Examples

Clearly, $\liminf s_n \leq \limsup s_n$.

Definition 9. *Let* $\{s_n\}$ *be a bounded sequence.* $\{s_n\}$ **oscillates** *if lim inf* $s_n <$ *lim sup* s_n*.*

Exercises 2.4

1. True – False. Justify your answer by citing a theorem, giving a proof, or giving a counter-example.

   (a) A sequence $\{s_n\}$ converges to s if and only if every subsequence of $\{s_n\}$ converges to s.

(b) Every bounded sequence is convergent.

(c) Let $\{s_n\}$ be a bounded sequence. If $\{s_n\}$ oscillates, then the set S of subsequential limits of $\{s_n\}$ has at least two points.

(d) Every sequence has a convergent subsequence.

(e) $\{s_n\}$ converges to s if and only if $\liminf s_n = \limsup s_n = s$.

2. Prove or give a counterexample.

(a) Every oscillating sequence has a convergent subsequence.

(b) Every oscillating sequence diverges.

(c) Every divergent sequence oscillates.

(d) Every bounded sequence has a Cauchy subsequence.

(e) Every monotone sequence has a bounded subsequence.