Math 6320 | Functions of a Real Variable | Fall 2015

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HOMEWORK 10

Due in class *Fri, Nov. 6.*

1. *Folland, 2.50.* Suppose $(X, \mathcal{M}, \mu)$ is a σ-finite measure space and $f \in L^+(X)$. Let

$$G_f = \{(x, y) \in X \times [0, \infty] : y \leq f(x)\}.$$

Then G_f is $\mathcal{M} \times \mathcal{B}_{\mathbb{R}}$-measurable and $\mu \times m(G_f) = \int f \, d\mu$; the same is also true if the inequality $y \leq f(x)$ in the definition of G_f is replaced by $y < f(x)$. (To show measurability of G_f, note that the map $(x, y) \mapsto f(x) - y$ is the composition of $(x, y) \mapsto (f(x), y)$ and $(z, y) \mapsto z - y$.) This is the definitive statement of the familiar theorem from calculus, "the integral of a function is the area under its graph."

2. *Folland, 2.51.* Let $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ be arbitrary measure spaces (not necessarily σ-finite).

(a) If $f : X \to \mathbb{C}$ is $\mathcal{M}$-measurable, $g : Y \to \mathbb{C}$ is $\mathcal{N}$-measurable, and $h(x, y) = f(x)g(y)$, then h is $\mathcal{M} \otimes \mathcal{N}$-measurable.

(b) If $f \in L^1(\mu)$ and $g \in L^1(\nu)$, then $h \in L^1(\mu \times \nu)$ and $\int h \, d(\mu \times \nu) = [\int f \, d\mu][\int g \, d\nu]$.

3. *Folland, 2.55.* Let $E = [0, 1] \times [0, 1]$. Investigate the existence and equality of $\int_E f \, dm^2$, $\int_0^1 \int_0^1 f(x, y) \, dx \, dy$, and $\int_0^1 \int_0^1 f(x, y) \, dy \, dx$ for the following f.

(a) $f(x, y) = (x^2 - y^2)(x^2 + y^2)^{-2}$.

(b) $f(x, y) = (1 - xy)^{-a}$ for some $a > 0$.

(c) $f(x, y) = \begin{cases} (x - \frac{1}{2})^{-3} & \text{if } 0 < y < |x - \frac{1}{2}|, \\ 0 & \text{otherwise.} \end{cases}$

4. *Folland, 2.59.* Let $f(x) = x^{-1} \sin x$.

(a) Show that $\int_0^\infty |f(x)| \, dx = \infty$.

(b) Show that $\lim_{b \to \infty} \int_0^b f(x) \, dx = \frac{1}{2}\pi$ by integrating $e^{-xy} \sin x$ with respect to x and y. (In view of part (a), some care is needed in passing to the limit as $b \to \infty$.)