Math 6320 | Functions of a Real Variable | Fall 2015
Dr. Vaughn Climenhaga, PGH 665

HOMEWORK 4

Due in class *Fri, Sep. 18.*

1. *Folland, 1.18.* Let $\mathcal{A} \subset \mathcal{P}(X)$ be an algebra, $\mathcal{A}_\sigma$ the collection of countable unions of sets in $\mathcal{A}$, and $\mathcal{A}_{\sigma\delta}$ the collection of countable intersections of sets in $\mathcal{A}_\sigma$. Let μ_0 be a premeasure on $\mathcal{A}$ and μ^* the induced outer measure.
 (a) For any $E \subset X$ and $\epsilon > 0$ there exists $A \in \mathcal{A}_\sigma$ with $E \subset A$ and $\mu^*(A) \le \mu^*(E) + \epsilon$.
 (b) If $\mu^*(E) < \infty$, then E is μ^*-measurable iff there exists $B \in \mathcal{A}_{\sigma\delta}$ with $E \subset B$ and $\mu^*(B \setminus E) = 0$.
 (c) If μ_0 is σ-finite, the restriction $\mu^*(E) < \infty$ in (b) is superfluous.

2. *Folland, 1.19.* Let μ^* be an outer measure on X induced from a finite premeasure μ_0. If $E \subset X$, define the **inner measure** of E to be $\mu_*(E) = \mu_0(X) - \mu^*(E^c)$. Then E is μ^*-measurable iff $\mu^*(E) = \mu_*(E)$. (Use the previous exercise.)

3. *Folland, 1.23.* Let $\mathcal{A}$ be the collection of finite unions of sets of the form $(a, b] \cap \mathbb{Q}$ where $-\infty \le a < b \le \infty$.
 (a) $\mathcal{A}$ is an algebra on $\mathbb{Q}$. (Use Proposition 1.7.)
 (b) The σ-algebra generated by $\mathcal{A}$ is $\mathcal{P}(\mathbb{Q})$.
 (c) Define μ_0 on $\mathcal{A}$ by $\mu_0(\emptyset) = 0$ and $\mu_0(A) = \infty$ for $A \ne \emptyset$. Then μ_0 is a premeasure on $\mathcal{A}$, and there is more than one measure on $\mathcal{P}(\mathbb{Q})$ whose restriction to $\mathcal{A}$ is μ_0.

4. *Folland, 1.30.* If $E \in \mathcal{L}$ and $m(E) > 0$, for any $\alpha < 1$ there is an open interval I such that $m(E \cap I) > \alpha m(I)$. *(Recall that m is Lebesgue measure and $\mathcal{L}$ is the collection of Lebesgue measurable sets.)*

5. *Folland, 1.32.* Suppose $\{\alpha_j\}_1^\infty \subset (0, 1)$.
 (a) $\prod_1^\infty (1 - \alpha_j) > 0$ iff $\sum_1^\infty \alpha_j < \infty$. (Compare $\sum_1^\infty \log(1 - \alpha_j)$ to $\sum \alpha_j$.)
 (b) Given $\beta \in (0, 1)$, exhibit a sequence $\{\alpha_j\}$ such that $\prod_1^\infty (1 - \alpha_j) = \beta$.