Math 6320 Functions of a Real Variable Fall 2015
Dr. Vaughn Climenhaga, PGH 665

HOMEWORK 5

Due in class *Fri, Sep. 25.*

1. *Folland, 1.28.* Let F be increasing and right continuous, and let μ_F be the associated measure. Then $\mu_F(\{a\}) = F(a) - F(a-)$, $\mu_F([a,b)) = F(b-) - F(a-)$, $\mu_F([a,b]) = F(b) - F(a-)$, and $\mu_F((a,b)) = F(b-) - F(a)$. *Here* $F(x-) := \lim_{y\to x^-} F(y)$.

2. *Folland, 1.31.* If $E \in \mathcal{L}$ and $m(E) > 0$, the set $E - E = \{x - y \mid x, y \in E\}$ contains an interval centered at 0. (If I is an open interval as in Exercise 30 with $\alpha > \frac{3}{4}$, so $m(E \cap I) > \alpha m(I)$, then $E - E$ contains $(-\frac{1}{2}m(I), \frac{1}{2}m(I))$.)

3. As in the lecture, let S be a finite set with d elements, and $X = S^{\mathbb{N}}$ the space of infinite sequences of symbols from S, with $d(x,y) = 2^{-\min\{n \mid x_n \neq y_n\}}$. Recall that given a word $w \in S^n$, we write $|w| = n$, and $[w] = \{x \in X \mid x_i = w_i \text{ for all } 1 \leq i \leq n\}$ for the *cylinder* defined by w.

The *shift map* on X is $\sigma\colon X \to X$ defined by $\sigma(x_1x_2x_3\cdots) = x_2x_3x_4\cdots$. A Borel measure μ on X is said to be *σ-invariant* if $\mu(\sigma^{-1}E) = \mu(E)$ for every Borel set $E \subset X$. Prove the following.

(a) For every probability vector p, the Bernoulli measure μ defined by $\mu[w] = \prod_{i=1}^{|w|} p_{w_i}$ is σ-invariant.

(b) If $x \in X$ is a periodic sequence (there is k such that $x_{n+k} = x_n$ for all $n \in \mathbb{N}$), then the measure $\mu = \frac{1}{k}\sum_{i=0}^{k-1} \delta_{\sigma^i x}$ is σ-invariant, where δ_y is the point mass at y defined by

$$\delta_y(E) = \begin{cases} 1 & \text{if } y \in E, \\ 0 & \text{otherwise.} \end{cases}$$

(c) Given a probability vector π and a $d \times d$ matrix P whose rows are probability vectors, recall that the Markov measure defined by (π, P) is given by $\mu[w] = \pi_{w_1} \prod_{i=1}^{|w|-1} P_{w_i w_{i+1}}$. Prove that this measure is σ-invariant if and only if π is a left eigenvector of P with eigenvalue 1; that is, iff $\pi P = \pi$.

4. Let X be as in the previous problem and let μ be a σ-invariant measure. Consider for each $n \in \mathbb{N}$ the number

$$H_n(\mu) := \sum_{w \in S^n} -\mu[w] \log \mu[w].$$

(a) Prove that $H_{m+n}(\mu) \leq H_m(\mu) + H_n(\mu)$ for every $m, n \in \mathbb{N}$; that is, the sequence $\{H_n(\mu)\}_n$ is *subadditive*.

(b) Prove that if a_n is any subadditive sequence ($a_{m+n} \leq a_m + a_n$), then $\lim_{n\to\infty} \frac{1}{n} a_n$ exists and is equal to $\inf_n \frac{1}{n} a_n$.

(c) The previous two parts show that the limit $\lim_{n\to\infty} \frac{1}{n} H_n(\mu)$ exists for any μ. Denote this limit by $h(\mu)$; this is called the *(Kolmogorov–Sinai) entropy* of μ. Prove that if μ is a Bernoulli measure given by a probability vector p, then $h(\mu) = \sum_{i=1}^{d} -p_i \log p_i$.