HOMEWORK 6

Due in class *Fri, Oct. 2.*

1. *Folland, 2.4.* If $f\colon X \to \overline{\mathbb{R}}$ and $f^{-1}((r,\infty]) \in \mathcal{M}$ for each $r \in \mathbb{Q}$, then f is measurable.

2. *Folland, 2.6.* The supremum of an uncountable family of measurable $\overline{\mathbb{R}}$-valued functions on X can fail to be measurable (unless the σ-algebra $\mathcal{M}$ is very special).

3. *Folland, 2.7.* Suppose that for each $\alpha \in \mathbb{R}$ we are given a set $E_\alpha \in \mathcal{M}$ such that $E_\alpha \subset E_\beta$ whenever $\alpha < \beta$, $\bigcup_{\alpha\in\mathbb{R}} E_\alpha = X$, and $\bigcap_{\alpha\in\mathbb{R}} E_\alpha = \emptyset$. Then there is a measurable function $f\colon X \to \mathbb{R}$ such that $f(x) \le \alpha$ on E_α and $f(x) \ge \alpha$ on E_α^c for every α. (Use Exercise 2.4.)

4. *Folland, 2.9.* Let $f\colon [0,1] \to [0,1]$ be the Cantor function (§1.5), and let $g(x) = f(x) + x$. *Recall that f is defined as follows: given $x \in C$ with $x = \sum_1^\infty a_j 3^{-j}$ for $a_j \in \{0,2\}$, let $f(x) = \sum_1^\infty \frac{a_j}{2} 2^{-j}$. This was referred to in the lecture as the 'devil's staircase'.*

(a) g is a bijection from $[0,1]$ to $[0,2]$, and $h = g^{-1}$ is continuous from $[0,2]$ to $[0,1]$.
(b) If C is the Cantor set, $m(g(C)) = 1$.
(c) Let $A \subset g(C)$ be a Lebesgue nonmeasurable set (such a set exists by Exercise 1.29). Let $B = g^{-1}(A)$. Then B is Lebesgue measurable but not Borel.
(d) There exist a Lebesgue measurable function F and a continuous function G on $\mathbb{R}$ such that $F \circ G$ is not Lebesgue measurable.