HOMEWORK 8

Due in class *Fri, Oct. 16.*

1. *Folland, 2.20.* (A generalized Dominated Convergence Theorem) If $f_n, g_n, f, g \in L^1$, $f_n \to f$ and $g_n \to g$ a.e., $|f_n| \leq g_n$, and $\int g_n \to \int g$, then $\int f_n \to \int f$. (Rework the proof of the dominated convergence theorem.)

2. *Folland, 2.21.* Suppose $f_n, f \in L^1$ and $f_n \to f$ a.e. Then $\int |f_n - f| \to 0$ iff $\int |f_n| \to \int |f|$. (Use the previous problem.)

3. *Folland, 2.22.* Let μ be counting measure on $\mathbb{N}$. Interpret Fatou's lemma and the monotone and dominated convergence theorems as statements about infinite series.

4. *Folland, 2.25.* Let $f(x) = x^{-1/2}$ if $0 < x < 1$, $f(x) = 0$ otherwise. Let $\{r_n\}_1^\infty$ be an enumeration of the rationals, and $g(x) = \sum_1^\infty 2^{-n} f(x - r_n)$.
(a) $g \in L^1(m)$, and in particular $g < \infty$ a.e.
(b) g is discontinuous at every point and unbounded on every interval, and it remains so after any modification on a Lebesgue null set.
(c) $g^2 < \infty$ a.e., but g^2 is not integrable on any interval.

5. *Folland, 2.28.* Compute the following limits and justify the calculations:
(a) $\lim_{n\to\infty} \int_0^\infty (1 + (x/n))^{-n} \sin(x/n)\, dx$.
(b) $\lim_{n\to\infty} \int_0^1 (1 + nx^2)(1 + x^2)^{-n}\, dx$.
(c) $\lim_{n\to\infty} \int_0^\infty n \sin(x/n)[x(1 + x^2)]^{-1}\, dx$.
(d) $\lim_{n\to\infty} \int_a^\infty n(1 + n^2x^2)^{-1}\, dx$. (The answer depends on whether $a > 0$, $a = 0$, or $a < 0$. How does this accord with the various convergence theorems?)