Math 7350 | Geometry of Manifolds | Spring 2015
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Review material

1. Summary of lectures

Some material may have been covered on a slightly different day than what is listed.

Chapter 1, Smooth manifolds

Lec 1, Jan 21. Basic definitions: manifolds, coordinate charts, transition maps, smooth atlases, smooth structure. Examples: $\mathbb{R}^n, S^n$.

Lec 2, Jan 23. Different smooth structures on $\mathbb{R}$, existence of smooth structures, direct products, more examples: $\mathbb{T}^n$, regular level sets via implicit function theorem.

Lec 3, Jan 26. More examples: $\mathbb{R}P^n, \mathbb{C}P^n$, Grassmannian.

Lec 4, Jan 28. Smooth structure on Grassmannian. Torus and projective space as quotient spaces. Manifolds with boundary.

Chapter 2, Smooth maps

Lec 5, Jan 30. Smooth maps, coordinate representations.

Lec 6, Feb 2. Examples of smooth maps $\mathbb{R} \to S^1$, $\mathbb{R}^n \to \mathbb{T}^n$. Linear map on $\mathbb{R}^2$ via $A \in SL(2,\mathbb{Z})$ descends to smooth map on $\mathbb{T}^2$. Diffeomorphisms, equivalence of smooth structures.

Lec 7, Feb 4. Cutoff functions, bump functions, partitions of unity.

Lec 8, Feb 6. Paracompactness, regular coordinate balls, existence of partitions of unity.

Lec 9, Feb 9. Group actions and quotient manifolds. (Not in book.)

Chapter 3, Tangent vectors

Lec 10, Feb 11. Geometric tangent vectors in $\mathbb{R}^n$. Tangent vectors as derivations.

Lec 11, Feb 13. Differential of a smooth map. Canonically identified tangent spaces: T_pU and T_pM when $U \subset M$ open; T_pV and V when V a linear map. Coordinate basis for T_pM coming from a chart.

Lec 12, Feb 16. Change of coordinates in T_pM. Tangent bundle. Differential in terms of pullback of smooth functions.

Lec 13, Feb 18. Velocity of curves, tangent vectors as equivalence classes of curves. Charts and transition maps for TM.

Lec 14, Feb 20. Categories and functors.

Surfaces (not in Lee)

Lec 15, Feb 23. 2-cell embeddings, triangulations, spheres with handles and cross caps.

Lec 16, Feb 25. Euler characteristic, Morse lemma, Morse functions.

Lec 17, Feb 27. More about Morse theory, classification of surfaces.

Chapter 4, Submersions, immersions, and embeddings

Lec 18, Mar 2. Rank of a smooth map, definition of submersions and immersions, local diffeomorphisms.

Lec 19, Mar 4. Maps of constant rank. Onto implies submersion, 1-1 implies immersion. Smooth embeddings.

Lec 20, Mar 6. Local sections, characterisation of submersions in terms of smooth local sections.

Lec 21, Mar 9. Smooth covering maps, universal covering manifold.

Chapter 5, Submanifolds

Lec 22, Mar 11. Embedded and immersed submanifolds. Local k-slice condition for embedded submanifolds.

Lec 23, Mar 23. Level sets of maps of constant rank are embedded submanifolds. Regular points and critical points.

Lec 24, Mar 25. Topology and smooth structure on embedded submanifold. Characterisation of T_pS.

Chapter 6, Sard's theorem

Lec 25, Mar 27. Sets of measure 0, Sard's theorem.

Lec 26, Mar 30. , Whitney embedding theorem.

Chapter 7, Lie groups

Lec 27, Apr 1. Definition of Lie group, left and right translation. Lie group homomorphisms have constant rank.

Lec 28, Apr 3. Universal covering group, examples. Lie subgroups.

Lec 29, Apr 6. Kernels of Lie group homomorphisms are embedded Lie subgroups. Examples: $SL(n,\mathbb{R}) \subset GL^+(n,\mathbb{R}) \subset GL(n,\mathbb{R})$, image of curve on $\mathbb{T}^2$ with irrational slope. Group actions and equivariant maps. Equivariant maps have constant rank.

Lec 30, Apr 8. Examples of equivariant maps. $O(n)$ is a level set of an equivariant map. Representations of Lie groups.

Chapter 8, Vector fields

Lec 31, Apr 10. Vector field is a section of the canonical submersion $TM \to M$. Space of smooth vector fields $\mathfrak{X}(M)$ is a vector space. Local and global frames. Vector fields as derivations on $C^\infty(M)$. Pushforward of a vector field under a diffeomorphism.

Lec 32, Apr 13. Lie bracket of two vector fields. Abstract definition of Lie algebra. Space of left-invariant vector fields on a Lie group is closed under Lie bracket. Lie algebra of a Lie group.

Lec 33, Apr 15. Isomorphism between $\mathrm{Lie}(G)$ and T_eG. Lie algebra of $GL(n,\mathbb{R})$. Lie group homomorphisms induce Lie algebra homomorphisms. Lie subalgebras.

Chapter 11, Cotangent bundle

Lec 34, Apr 17. Covectors, dual basis, cotangent space. Cotangent bundle, covector field, differential 1-form. f a smooth function $\Rightarrow df$ is a 1-form.

Lec 35, Apr 20. Exact 1-forms, pullbacks. Line integrals of 1-forms. Fundamental theorem for line integrals. Closed 1-forms. Exact implies closed but not (always) vice versa.

Chapters 12–16, Tensors, Riemannian metrics, Differential forms, Orientations, Integration on manifolds

Lec 36, Apr 22. Multilinear functionals, symmetric and alternating tensors. Tensor product. Tensor bundle, tensor fields. Pullbacks of tensor fields.

Lec 37, Apr 24. Riemannian metrics. Musical isomorphisms.

Lec 38, Apr 27. Exterior forms (alternating k-tensors). Differential k-forms on manifolds. A basis for $\Lambda^k V^*$.

Lec 39, Apr 29. Wedge product. Exterior derivative.

Lec 40, May 1. Integrating k-forms. Stokes theorem.

Lec 41, May 4. Relationship between exterior derivative and gradient, curl, divergence; relationship between Stokes theorem and vector calculus.

2. Detailed summary of last 6 lectures

Goal is to explain following diagram, why it commutes, and how it unifies vector calculus

$$\begin{tikzcd}
C^\infty(\mathbb{R}^3) \arrow[r, "\mathrm{grad}"] \arrow[d, "="] & \mathfrak{X}(\mathbb{R}^3) \arrow[r, "\mathrm{curl}"] \arrow[d, "\flat"] & \mathfrak{X}(\mathbb{R}^3) \arrow[r, "\mathrm{div}"] \arrow[d, "\beta = * \circ \flat"] & C^\infty(\mathbb{R}^3) \arrow[d, "*"] \\
\Omega^0(\mathbb{R}^3) \arrow[r, "d"] & \Omega^1(\mathbb{R}^3) \arrow[r, "d"] & \Omega^2(\mathbb{R}^3) \arrow[r, "d"] & \Omega^3(\mathbb{R}^3)
\end{tikzcd}$$

2.1. **Tensors.** V a vector space, V^* its dual space.

$V^* = \{\omega\colon V \to \mathbb{R} \mid \omega \text{ linear}\}$.

$T^k V^* = \{\omega\colon \overbrace{V \times \cdots \times V}^{k \text{ times}} \to \mathbb{R} \mid \omega \text{ multilinear}\}$

Elements of T^kV^* are (covariant) k-tensors. k is the rank, or degree, of the tensor. Inner product is a 2-tensor. Determinant for $n \times n$ matrices takes n vectors as the columns of the matrix and becomes an n-tensor.

Given a permutation $\sigma \in S_k$ write ${}^{\sigma}\omega(v_1, \dots, v_k) = \omega(v_{\sigma(1)}, \dots, v_{\sigma_k})$. Inner product is symmetric: ${}^{\sigma}\omega = \omega$ for all σ. Determinant is alternating: ${}^{\sigma}\omega = (\operatorname{sgn}\sigma)\omega$ for all σ.

Tensor product of $\omega \in T^kV^*$ and $\eta \in T^{\ell}V^*$ is $\omega \otimes \eta \in T^{k+\ell}V^*$ with

$$\omega \otimes \eta(v_1, \dots, v_k, u_1, \dots, u_\ell) = \omega(v_1, \dots, v_k)\eta(u_1, \dots, u_\ell).$$

If e^1, e^2 is dual basis for $(\mathbb{R}^2)^*$ then $\det = e^1 \otimes e^2 - e^2 \otimes e^1$

Theorem: $\{e^{i_1} \otimes \cdots \otimes e^{i_k} \mid i_j \in \{1, \dots, n\}\}$ is a basis for $T^k\mathbb{R}^n$

$T^kT^*M = \bigsqcup_{p \in M} T^k(T_p^*M)$ is the bundle of k-tensors

$\mathcal{T}^k(M) = \Gamma(T^kT^*M)$ is space of sections of this bundle, ie. smooth (covariant) k-tensor fields

2.2. **Pullbacks, metrics, forms.** 1-tensor fields are covector fields. All tensor fields pull back just as covector fields do. Given $F\colon M \to N$ smooth and $\omega \in \mathcal{T}^k(N)$, define $F^*\omega \in \mathcal{T}^k(M)$ by

$$F^*\omega(v_1, \dots, v_k) = \omega(dF_p(v_1), \dots, dF_p(v_k)) \text{ for } v_i \in T_pM$$

Riemannian metric is smooth symmetric 2-tensor field such that every $g_p\colon (T_pM) \times (T_pM) \to \mathbb{R}$ is positive definite. Amounts to attaching an inner product to every tangent space T_pM in a smooth way. Gives an isomorphism between T_pM and T_p^*M by

$$v^\flat(w) = \langle v, w\rangle \text{ for } v \in T_p^M.$$

Similarly given $\omega \in T_p^*M$, $\omega^\sharp \in T_pM$ is uniquely defined by $\langle \omega^\sharp, v\rangle = \omega(v)$ for all $v \in T_pM$. These are the musical isomorphisms: they raise and lower indices when ω, v are written as linear combinations $\omega_i dx^i$ and $v^i \frac{\partial}{\partial x^i}$.

Alternating k-tensors on V form a subspace of T^kV^*, denoted Λ^kV^*.

$\alpha \in T^kV^*$ is alternating iff it sends linearly dependent sets to 0, iff it sends $v_1, \dots, v_k$ to 0 whenever $v_i = v_j$ for some $i \neq j$. Suggests a relationship to determinant

Bundle of alternating k-tensors is $\Lambda^kT^*M = \bigsqcup_{p \in M} \Lambda^k(T_p^*M)$, space of sections is $\Omega^k(M) = \Gamma(\Lambda^kT^*M)$, the space of differential k-forms.

2.3. **Basis for Λ^kV^*.** Toy example: $\Lambda^2(\mathbb{R}^2)^*$. Everything in $T^2(\mathbb{R}^2)^*$ can be written as $\omega_{ij}e^i \otimes e^j$. Note that ${}^{\sigma}\omega = \omega_{ij}e^j \otimes e^i = \omega_{ji}e^i \otimes e^j$, and so ω is alternating iff $\omega_{ij} = -\omega_{ji}$ for all i, j. Thus $\omega_{11} = \omega_{22} = 0$ and $\omega_{12} = -\omega_{21}$. Thus ω is a scalar multiple of $e^1 \otimes e^2 - e^2 \otimes e^1$.

More generally, with $V = \mathbb{R}^n$ and $I = (i_1, \dots, i_k)$ a multiindex (each $i_j \in \{1, \dots, n\}$), write $\alpha^I = e^{i_1} \otimes \cdots \otimes e^{i_k}$ and write $\omega \in T^k V^*$ as $\omega = \omega_I \alpha^I$. (Sum over all multiindices.) Given $\sigma \in S_k$ write $I_\sigma = (i_{\sigma(1)}, \dots, i_{\sigma(k)})$. Then

$$^\sigma\omega = \omega_I \alpha^{I_\sigma} = \omega_{J_\sigma} \alpha^J,$$

and comparing coefficients we see that ω is alternating iff $\omega_{I_\sigma} = (\operatorname{sgn}\sigma)\omega_I$ for all I, σ.

In particular, if I has a repeated index then $\omega_I = 0$. Moreover, ω_I determines ω_{I_σ} for all $\sigma \in S_k$. So it suffices to specify ω_I when I is increasing: $i_1 < \cdots < i_k$. Write

$$e^I = e^{i_1} \wedge \cdots \wedge e^{i_k} := \sum_{\sigma \in S_k} (\operatorname{sgn}\sigma) \alpha^{I_\sigma} = \sum_{\sigma \in S_k} (\operatorname{sgn}\sigma) e^{i_{\sigma(1)}} \otimes \cdots \otimes e^{i_{\sigma(k)}}$$

then $\{e^I \mid I \text{ an increasing } k\text{-multiindex}\}$ is a basis for $\Lambda^k V^*$. Note that this means dimension of $\Lambda^k V^*$ is $\binom{n}{k}$

For now $e^i \wedge e^j = e^{(i,j)}$ is just notation. The wedge product will be introduced later.

Examples: $e^{(1,2)} = e^1 \otimes e^2 - e^2 \otimes e^1$ on $\mathbb{R}^2$ in the 2×2 determinant.

In $\mathbb{R}^3$, get $e^{(1,2,3)}(v, w, x) = \sum_{\sigma \in S_3} (\operatorname{sgn}\sigma) v^{\sigma(1)} w^{\sigma(2)} x^{\sigma(3)} = \det[v \mid w \mid x]$

More generally, $e^I(v_1, \dots, v_k)$ is the determinant of the $k \times k$ matrix with entries $e^{i_j}(v_{j'})$, where $j, j' \in \{1, 2, \dots, k\}$.

Now we see that $\Lambda^n V^*$ is one-dimensional when $n = \dim V$, and if $e^1, \dots, e^n$ is a basis for V^* then $e^1 \wedge \cdots \wedge e^n$ is a basis (with one element) for $\Lambda^n V^*$.

We translate all of the above into $\Omega^k(M)$ by noting that in any local coordinate chart we have a local frame for T^*M given by $dx^1, \dots, dx^n$, and so we use dx^i in place of e^i in the above. Thus locally, any 1-form is $\omega_i\, dx^i$, where ω_i are smooth functions, similarly 2-forms are $\omega_I\, dx^{i_1} \wedge dx^{i_2}$ where $I = (i_1, i_2)$ ranges over all $i_1 < i_2$, and so on.

In particular, given any $\omega \in \Omega^3(\mathbb{R}^3)$, there is a smooth function $f \in C^\infty(\mathbb{R}^3)$ such that $\omega_p = f(p)\, dx^1 \wedge dx^2 \wedge dx^3$ for every $p \in \mathbb{R}^3$. We write $*$ for this correspondence $\Omega^3(\mathbb{R}^3) \to C^\infty(\mathbb{R}^3)$ and for its inverse.

Now we have completed all but one of the vertical arrows in the motivating $\mathbb{R}^3$ diagram. It remains to explain $\beta\colon \mathfrak{X}(\mathbb{R}^3) \to \Omega^2(\mathbb{R}^3)$.

Given $v \in V$ and $\omega \in \Lambda^k V^*$ we can define $i_v\omega \in \Lambda^{k-1} V^*$ by

$$(i_v\omega)(v_1, \dots, v_k) = \omega(v, v_1, \dots, v_{k-1}).$$

This is the interior product of v into ω. Now given $X \in \mathfrak{X}(\mathbb{R}^3)$ we define $\beta(X) \in \Omega^2(\mathbb{R}^3)$ by

$$\beta(X)_p = i_{X_p}(dx^1 \wedge dx^2 \wedge dx^3)_p,$$

that is, $\beta(X)$ is the 2-form that acts on vectors $v, w \in T_p\mathbb{R}^3$ by

$$\beta(X)(v, w) = (dx^1 \wedge dx^2 \wedge dx^3)(X_p, v, w) = \det[X_p \mid v \mid w].$$

We can write β quite explicitly now. When $X = \frac{\partial}{\partial x^1}$ we get

$$\begin{aligned}\beta(X)(v,w) = \det\begin{pmatrix}1 & v^1 & w^1\\ 0 & v^2 & w^2\\ 0 & v^3 & w^3\end{pmatrix} &= v^2w^3 - v^3w^2\\ &= (dx^2\otimes dx^3)(v,w) - (dx^3\otimes dx^2)(v,w)\\ &= (dx^2\wedge dx^3)(v,w).\end{aligned}$$

Thus $\beta\colon \frac{\partial}{\partial x^1}\mapsto dx^2\wedge dx^3$, with similar formulas for the other coordinate vector fields.

2.4. **Wedge product and exterior derivative.** The symbol $\wedge$ in $e^{i_1}\wedge\cdots\wedge e^{i_k}$ can be given a more meaningful interpretation. Assume V has been given a basis (E_i) and (e^i) is the corresponding dual basis. We define a bilinear map

$$\wedge\colon \Lambda^kV^*\times\Lambda^\ell V^*\to\Lambda^{k+\ell}V^*$$

by first specifying it on the basis elements in the natural way: $e^Ie^J = e^{IJ}$, that is

$$\left(e^{i_1}\wedge\cdots\wedge e^{i_k}\right)\wedge\left(e^{j_1}\wedge\cdots\wedge e^{j_\ell}\right) = e^{i_1}\wedge\cdots\wedge e^{i_k}\wedge e^{j_1}\wedge\cdots\wedge e^{j_\ell}$$

Notice that the product is zero if the multiindices I and J share any entries. Now we extend to the full wedge product: given $\omega\in\Lambda^kV^*$ and $\eta\in\Lambda^\ell V^*$ we define $\omega\wedge\eta\in\Lambda^{k+\ell}V^*$ by

$$\omega\wedge\eta = (\omega_Ie^I)\wedge(\eta_Je^J) = \omega_I\eta_Je^{IJ},$$

where the sum is over all k-multiindices I and all ℓ-multiindices J.

The exterior derivative $d\colon\Omega^k(M)\to\Omega^{k+1}(M)$ is similarly defined first for "simple elements" and then by taking sums. We give a definition in local coordinates – other definitions are available but require some more work, and this definition will be the most useful for our work in $\mathbb{R}^3$. Fix local coordinates (U,φ) so that (dx^i) is a frame for T^*U. Given $f\in C^\infty(U)$ and a k-multiindex J, note that $f\,dx^J$ is a k-form on U, and define

$$d(f\,dx^J) = df\wedge dx^J = \frac{\partial f}{\partial x^i}\,dx^i\wedge dx^J = \frac{\partial f}{\partial x^i}\,dx^i\wedge dx^{j_1}\wedge\cdots\wedge dx^{j_k}.$$

This is a $(k+1)$-form on U. Extend this to all $\Omega^k(U)$ by

$$d\omega = d(\omega_J\,dx^J) = (d\omega_J)\wedge(dx^J) = \frac{\partial\omega_J}{\partial x^i}\,dx^i\wedge dx^{j_1}\wedge\cdots\wedge dx^{j_k}.$$

In the specific case $k=1$, we have $\omega\in\Omega^1(U)$ written as $\omega=\omega_j\,dx^j$, and get $d\omega\in\Omega^2(U)$ given by

$$d\omega = d\omega_j\wedge dx^j = \frac{\partial\omega_j}{\partial x^i}\,dx^i\wedge dx^j$$

where the sum is over all i, j. If we agree to take a sum only over $i < j$, then this can be rewritten as

$$d\omega = \sum_{i<j} \left(\frac{\partial \omega_j}{\partial x^i} - \frac{\partial \omega_j}{\partial x^i} \right) dx^i \wedge dx^j,$$

and we see from the chapter on covector fields that ω is closed if and only if $d\omega = 0$.

We make this a general definition. A k-form ω is exact if $\omega = d\eta$ for some $\eta \in \Omega^{k-1}$, and ω is closed if $d\omega = 0$.

Theorem: exact implies closed. First one proves that d satisfies a version of the product rule (we did not do this explicitly in class but it is not too hard):

$$d(\omega \wedge \eta) = (d\omega) \wedge \eta + (-1)^k \omega \wedge (d\eta),$$

where k is the rank of ω. Then $d^2 = 0$ is a computation: $d\omega$ is a sum of terms of the form $df \wedge dx^J$, and thus to show $d(d\omega) = 0$ it suffices to show that $d(df \wedge dx^J) = 0$. We observe that

$$d(df \wedge dx^J) = d(df) \wedge dx^J - df \wedge d(dx^J)$$

and that $d(df) = 0$ from the chapter on covector fields. Finally, again by the product rule

$$d(dx^J) = d(dx^{j_1} \wedge \cdots \wedge dx^{j_k}) = \sum_{i=1}^{k} (-1)^{i-1} dx^{j_1} \wedge \cdots \wedge d(dx^{j_i}) \wedge \cdots dx^{j_k},$$

which vanishes because $d(dx^j) = 0$ for all j.

In fact, $d \colon \Omega^k(M) \to \Omega^{k+1}(M)$ is uniquely defined by the following properties:

(1) d is linear over $\mathbb{R}$;
(2) $d(\omega \wedge \eta) = (d\omega) \wedge \eta + (-1)^k \omega \wedge (d\eta)$;
(3) $d \circ d = 0$;
(4) given $f \in C^\infty(M) = \Omega^0(M)$, we have $df(X) = X(f)$ for every $X \in \mathfrak{X}(M)$.

Now we can see that the main $\mathbb{R}^3$ diagram from before commutes. First start with $f \in C^\infty(\mathbb{R}^3) = \Omega^0(\mathbb{R}^3)$ and observe that

$$df = \frac{\partial f}{\partial x^i} dx^i \quad \Rightarrow \quad (df)^\sharp = \sum_i \frac{\partial f}{\partial x^i} \frac{\partial}{\partial x^i} = \operatorname{grad}(f)$$

For the second square we show that given $Y \in \mathfrak{X}(\mathbb{R}^3)$ we have $\beta^{-1}(dY^\flat) = \operatorname{curl}(Y)$. First we observe that the correspondence $*$ between $C^\infty(\mathbb{R}^3) = \Omega^0(\mathbb{R}^3)$ and $\Omega^3(\mathbb{R}^3)$ extends more generally to a correspondence between $\Omega^k(\mathbb{R}^n)$ and $\Omega^{n-k}(\mathbb{R}^n)$ by first working with $\Lambda^k(\mathbb{R}^n)^*$ and $\Lambda^{n-k}(\mathbb{R}^n)^*$ and putting

$$*(e^I) = \operatorname{sgn}(I, I^c) e^{I^c}$$

for each k-multiindex I, where $I^c = \{1, \dots, n\} \setminus I$ is the multiindex with all $n - k$ entries that do not appear in I, arranged in increasing order, and (I, I^c) is the permutation of $\{1, \ldots, n\}$ obtained by listing first the elements of I and then those of I^c. This extends linearly to a map

$$*\colon \Lambda^k(\mathbb{R}^n)^* \to \Lambda^{n-k}(\mathbb{R}^n)^*.$$

Now in $\mathbb{R}^3$ we can use (dx^1, dx^2, dx^3) for the basis (e^1, e^2, e^3) and we get a correspondence $*$ between $\Omega^1(\mathbb{R}^3)$ and $\Omega^2(\mathbb{R}^3)$. (With $k = 0$ and $n = 3$ we recover the earlier definition of $*$.) This correspondence has $*(dx^1) = dx^2 \wedge dx^3$ and so on.

This is an example of the Hodge star operator, which can be defined on any Riemannian manifold. (Just as with the musical isomorphisms.)

Now we recall that the isomorphism $\beta\colon \mathfrak{X}(\mathbb{R}^3) \to \Omega^2(\mathbb{R}^3)$ takes $\frac{\partial}{\partial x^1}$ to $dx^2 \wedge dx^3$, and similarly with the other two coordinate vector fields. Observing that $Y^\flat = \sum_i Y^i dx^i \in \Omega^1(\mathbb{R}^3)$ and using the isomorphism $*\colon \Omega^1(\mathbb{R}^3) \to \Omega^2(\mathbb{R}^3)$ as just described, we see that $*(Y^\flat) = \beta(Y)$. In particular, given $\omega \in \Omega^2(\mathbb{R}^3)$ we have $\beta^{-1}(\omega) = (*\omega)^\sharp$.

So we take $Y \in \mathfrak{X}(\mathbb{R}^3)$ and show that $\beta^{-1}(dY^\flat) = (*dY^\flat)^\sharp = \operatorname{curl}(Y)$.

First, $Y^\flat = \sum_i Y^i dx^i = Y^1 dx^1 + Y^2 dx^2 + Y^3 dx^3$. To compute $dY^\flat$ we show what happens for the first term.

$$\begin{aligned} d(Y^1\, dx^1) = dY^1 \wedge dx^1 &= \frac{\partial Y^1}{\partial x^1}\, dx^1 \wedge dx^1 + \frac{\partial Y^1}{\partial x^2}\, dx^2 \wedge dx^1 + \frac{\partial Y^1}{\partial x^3}\, dx^3 \wedge dx^1 \\ &= -\frac{\partial Y^1}{\partial x^2}\, dx^1 \wedge dx^2 - \frac{\partial Y^1}{\partial x^3}\, dx^1 \wedge dx^3 \end{aligned}$$

where we use the fact that $\omega \wedge \omega = 0$ always, and that for 1-forms we have $\omega \wedge \eta = -\eta \wedge \omega$. (The change in sign, or lack thereof, depends on the rank of the forms involved.) Similar expressions come for the second and third terms $Y^i dx^i$, and we get

$$dY^\flat = \left(\frac{\partial Y^2}{\partial x^1} - \frac{\partial Y^1}{\partial x^2}\right) dx^1 \wedge dx^2 + \left(\frac{\partial Y^3}{\partial x^1} - \frac{\partial Y^1}{\partial x^3}\right) dx^1 \wedge dx^3 + \left(\frac{\partial Y^3}{\partial x^2} - \frac{\partial Y^2}{\partial x^3}\right) dx^2 \wedge dx^3$$

Applying $*$, we get that $*dY^\flat$ has the same form but with $dx^1 \wedge dx^2$ replaced by dx^3, and similarly for the other terms:

$$*dY^\flat = \left(\frac{\partial Y^2}{\partial x^1} - \frac{\partial Y^1}{\partial x^2}\right) dx^3 - \left(\frac{\partial Y^3}{\partial x^1} - \frac{\partial Y^1}{\partial x^3}\right) dx^2 + \left(\frac{\partial Y^3}{\partial x^2} - \frac{\partial Y^2}{\partial x^3}\right) dx^1$$

Finally, applying the musical isomorphism $\sharp$ yields the curl of Y:

$$(*dY^\flat)^\sharp = \left(\frac{\partial Y^2}{\partial x^1} - \frac{\partial Y^1}{\partial x^2}\right) \frac{\partial}{\partial x^3} - \left(\frac{\partial Y^3}{\partial x^1} - \frac{\partial Y^1}{\partial x^3}\right) \frac{\partial}{\partial x^2} + \left(\frac{\partial Y^3}{\partial x^2} - \frac{\partial Y^2}{\partial x^3}\right) \frac{\partial}{\partial x^1}$$

The story for divergence is similar but simpler. Given $Y \in \mathfrak{X}(\mathbb{R}^3)$ we want to show that $*(d\beta(Y)) = *d*Y^\flat = \operatorname{div}(Y)$. As before $Y^\flat = Y^1 dx^1 + Y^2 dx^2 + Y^3 dx^3$ and so

$$*Y^\flat = Y^1\, dx^2 \wedge dx^3 - Y^2\, dx^1 \wedge dx^3 + Y^3\, dx^1 \wedge dx^2.$$

The exterior derivative of the first term is

$$d(Y^1\,dx^2 \wedge dx^3) = dY^1 \wedge dx^2 \wedge dx^3 = \frac{\partial Y^1}{\partial x^i}\,dx^i \wedge dx^2 \wedge dx^3 = \frac{\partial Y^1}{\partial x^1}\,dx^1 \wedge dx^2 \wedge dx^3,$$

where we use the fact that wedge products of repeated terms vanish. Similarly we get

$$d(Y^2\,dx^1 \wedge dx^3) = \frac{\partial Y^2}{\partial x^2}\,dx^2 \wedge dx^1 \wedge dx^3 = -\frac{\partial Y^2}{\partial x^2}\,dx^1 \wedge dx^2 \wedge dx^3$$

and

$$d(Y^3\,dx^1 \wedge dx^2) = \frac{\partial Y^3}{\partial x^3}\,dx^3 \wedge dx^1 \wedge dx^2 = \frac{\partial Y^3}{\partial x^3}\,dx^1 \wedge dx^2 \wedge dx^3,$$

so that

$$d * Y^\flat = \left(\frac{\partial Y^1}{\partial x^1} + \frac{\partial Y^2}{\partial x^2} + \frac{\partial Y^3}{\partial x^3}\right) dx^1 \wedge dx^2 \wedge dx^3.$$

Applying $*$ once more we get

$$*d * Y^\flat = \frac{\partial Y^1}{\partial x^1} + \frac{\partial Y^2}{\partial x^2} + \frac{\partial Y^3}{\partial x^3} = \operatorname{div}(Y)$$

so the diagram commutes.

2.5. **Integration and Stokes theorem.** In a sense, 1-forms are “things that can be integrated over curves”. Similarly, k-forms are “things that can be integrated over k-dimensional submanifolds”.

First we integrate n-forms on open subsets of $\mathbb{R}^n$. Let $U \subset \mathbb{R}^n$ be open and let $\omega \in \Omega^n(U)$ be compactly supported in U. Then $f = *\omega \in C^\infty(U)$, that is, $\omega_p = f(p)\,dx^1 \wedge \cdots \wedge dx^n$. By definition,

$$\int_U \omega = \int_U f\,dV = \int_U f(p)\,dx^1 \cdots dx^n.$$

As with 1-forms on $\mathbb{R}$, this is invariant under orientation-preserving diffeomorphisms: if $U, V \subset \mathbb{R}^n$ are open and $G\colon U \to V$ is a diffeomorphism with $\det(DG) > 0$ at every point, then

$$\int_U G^*\omega = \int_V \omega,$$

where we recall that $G^*\omega$ is the pullback defined by

$$G^*\omega(v_1, \ldots, v_n)_p = \omega(dG_p(v_1), \ldots, dG_p(v_n))$$

so $G^*\colon \Omega^n(V) \to \Omega^n(U)$.

A manifold M is orientable if it has a smooth atlas such that all transition maps have positive Jacobian determinant. From now on we work only with charts chosen from such an atlas. Given such a chart (U, φ) and $\omega \in \Omega^n(M)$ supported in U, we define

$$\int_M \omega = \int_{\varphi(U)} (\varphi^{-1})^*\omega,$$

where we note that $\varphi^{-1}\colon \varphi(U) \to U$ is a diffeomorphism between an open set in $\mathbb{R}^n$ and the open set U, so that the pullback of ω under this map is an n-form on $\mathbb{R}^n$. We can extend this definition to arbitrary forms $\omega \in \Omega^n(M)$ by using a partition of unity subordinate to an oriented atlas and writing

$$\int_M \omega = \sum_i \int_M \psi_i \omega,$$

where we note that each $\psi_i \omega$ is compactly supported in some coordinate domain. Proposition 16.8 in Lee's book gives a way to compute integrals via parametrisations, without using partitions of unity.

Note that if M is a submanifold of a manifold with dimension $N > n$, and ω is any n-form on this larger manifold, then ω restricts to an n-form on M in a natural way, and thus can be integrated.

The key result is Stokes theorem. If M is an oriented smooth n-manifold with boundary, then its boundary ∂M inherits an orientation in a natural way, and given a compactly supported $\omega \in \Omega^{n-1}(M)$, we have

$$\int_M d\omega = \int_{\partial M} \omega.$$

Special cases: integrating an exact form over a manifold without boundary gives $\int_M d\omega = 0$, and integrating a closed form over a boundary gives $\int_{\partial M} \omega = 0$.

Several cases of this theorem are familiar from vector calculus.

Green's theorem. Given an open subset $D \subset \mathbb{R}^2$ and $P, Q \in C^\infty(D)$, writing $\omega = P\,dx + Q\,dy$ gives

$$\int_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial Y} \right) dx\,dy = \int_{\partial D} P\,dx + Q\,dy.$$

Divergence theorem. Given an open subset $U \subset \mathbb{R}^3$ and a vector field $X \in \mathfrak{X}(U)$, we have

$$\int_U \operatorname{div}(X)\,dV = \int_{\partial U} \langle X, N \rangle\,dA,$$

where dV is a volume element, dA is a surface area element, and N is the normal vector field to the surface ∂U.

To get this from Stokes theorem above, use the fact that $\operatorname{div}(X)\,dV = d(\beta(X))$, and

$$\int_U d(\beta(X)) = \int_{\partial U} \beta(X).$$

Then one needs to argue geometrically that $\beta(X) = X^1\,dx^2 \wedge dx^3 + \cdots$ is equal to the integrand on the right-hand side of the divergence theorem.

Finally, if $S \subset \mathbb{R}^3$ is a 2-dimensional surface and X is a vector field in $\mathbb{R}^3$, then

$$\int_S \langle \mathrm{curl}(X), N \rangle \, dA = \int_{\partial S} \langle X, T \rangle \, ds,$$

where N is a normal vector, dA is an area element, T is a tangent vector to the curve ∂S, and ds is length along that curve.

To get this from Stokes theorem one argues that $X^\flat = \langle X, T \rangle ds$ and that $d(X^\flat)$ gives the integrand on the left.