HOMEWORK 1

Due in class *Wednesday, Jan. 28.*

1. *(Lee, Problem 1-4).* **Locally finite covers**
Let M be a topological manifold, and let $\mathcal{U}$ be an open cover of M.
(a) Suppose each set in $\mathcal{U}$ intersects only finitely many others. Show that $\mathcal{U}$ is locally finite – that is, every point of M has a neighbourhood that intersects at most finitely many of the sets in $\mathcal{U}$.
(b) Give an example showing that the converse may be false.
(c) Show that the converse is true if the elements of $\mathcal{U}$ are precompact (have compact closures).

2. *(Lee, Problem 1-6).* **Distinct smooth structures**
Let M be a nonempty topological manifold of dimension $n \geq 1$. If M has a smooth structure, show that it has uncountably many distinct ones. [Hint: first show that for any $s > 0$, $F_s(x) = |x|^{s-1}x$ defines a homeomorphism from the unit ball in $\mathbb{R}^n$ to itself, which is a diffeomorphism if and only if $s = 1$.]

3. *(Lee, Problem 1-7).* **Stereographic coordinates**
Let $N = (0, \dots, 0, 1) \in S^n \subset \mathbb{R}^{n+1}$ be the north pole, and $S = (0, \dots, 0, -1)$ the south pole. The stereographic projection from the north pole is $\sigma \colon S^n \setminus \{N\} \to \mathbb{R}^n$ given by

$$\sigma(x^1, \dots, x^{n+1}) = \frac{(x^1, \dots, x^n)}{1 - x^{n+1}}.$$

The stereographic projection from the south pole is $\tilde{\sigma}(x) = -\sigma(-x)$ for $x \in S^n \setminus \{S\}$.
(a) Given $x \in S^n \setminus \{N\}$, show that $(\sigma(x), 0)$ is the point where the line in $\mathbb{R}^{n+1}$ through N and x intersects the linear subspace $x^{n+1} = 0$. Prove a similar result for $\tilde{\sigma}(x)$ when $x \in S^n \setminus \{S\}$.
(b) Show that σ is bijective, with

$$\sigma^{-1}(u^1, \dots, u^n) = \frac{(2u^1, \dots, 2u^n, |u|^2 - 1)}{|u|^2 + 1}$$

(c) Compute the transition map $\tilde{\sigma} \circ \sigma^{-1}$ and verify that the atlas consisting of the two stereographic projection charts defines a smooth structure on S^n.
(d) Show that this smooth structure is the same as the one defined in lecture (Example 1.31 in the book).

4. *(Lee, Problem 1-9).* **Complex projective n-space**
Complex projective n-space, denoted by $\mathbb{C}P^n$, is the set of all 1-dimensional complex-linear subspaces of $\mathbb{C}^{n+1}$, with the quotient topology inherited from the natural projection $\pi\colon \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{C}P^n$. Show that $\mathbb{C}P^n$ is a compact $2n$-dimensional topological manifold, and show how to give it a smooth structure analogous to the one we constructed for $\mathbb{R}P^n$. Note that we identify $\mathbb{C}^{n+1}$ with $\mathbb{R}^{2n+2}$ via the correspondence

$$(x^1 + iy^1, \dots, x^{n+1} + iy^{n+1}) \leftrightarrow (x^1, y^1, \dots, x^{n+1}, y^{n+1}).$$

5. *(Lee, Exercise 1.18).* **Compatibility of atlases**
Let M be a topological manifold. Show that two smooth atlases for M determine the same smooth structure if and only if their union is a smooth atlas.