Math 7350 | Geometry of Manifolds | Spring 2015

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HOMEWORK 2

Due in class *Wednesday, Feb. 4*.

1. More on Grassmannians

Let V be a n-dimensional real vector space and recall that given an integer $1 \leq k \leq n$, $G_k(V)$ is the Grassmann manifold whose elements are all the k-dimensional subspaces of V.

(a) We have seen that $G_k(V)$ is a smooth manifold for each k. Prove that it is compact.

(b) Prove that $G_k(V)$ and $G_{n-k}(V)$ are diffeomorphic.

2. *(Lee, Problem 2-8).* Define $F \colon \mathbb{R}^n \to \mathbb{R}P^n$ by $F(x^1, \ldots, x^n) = [x^1, \ldots, x^n, 1]$. Show that F is a diffeomorphism onto a dense open subset of $\mathbb{R}P^n$. Do the same for $G \colon \mathbb{C}^n \to \mathbb{C}P^n$ defined by $G(z^1, \ldots, z^n) = [z^1, \ldots, z^n, 1]$.

3. *(Lee, Problem 2-9).* Let p be a non-zero polynomial in one variable with complex coefficients. Show that there is a unique continuous map $\tilde{p} \colon \mathbb{C}P^1 \to \mathbb{C}P^1$ such that the following diagram commutes, where G is as in the previous problem.

$$\begin{tikzcd} \mathbb{C} \arrow[r, "G"] \arrow[d, "p"] & \mathbb{C}P^1 \arrow[d, "\tilde{p}"] \\ \mathbb{C} \arrow[r, "G"] & \mathbb{C}P^1 \end{tikzcd}$$

Show that the map $\tilde{p}$ is smooth. Is it a diffeomorphism?

4. *(Lee, Problem 2-10).* **Smoothness via functions**

For any topological space M, let $C(M)$ denote the algebra of continuous functions $f \colon M \to \mathbb{R}$. Given a continuous map $F \colon M \to N$, define $F^* \colon C(N) \to C(M)$ by $F^*(f) = f \circ F$.

(a) Show that F^* is a linear map.

(b) Suppose M and N are smooth manifolds. Show that $F \colon M \to N$ is smooth if and only $F^*(C^\infty(N)) \subset C^\infty(M)$.

(c) Suppose $F \colon M \to N$ is a homeomorphism between smooth manifolds. Show that it is a diffeomorphism iff F^* restricts to an isomorphism from $C^\infty(N)$ to $C^\infty(M)$.

5. Tori as quotients by lattices

Let $v, w \in \mathbb{R}^2$ be independent (hence a basis), and let $\Lambda = v\mathbb{Z} + w\mathbb{Z} = \{av + bw \mid a, b \in \mathbb{Z}\}$. We say that $\Lambda \subset \mathbb{R}^2$ is a **lattice**. The lattice Λ induces an equivalence relation on $\mathbb{R}^2$ by putting $x \sim y$ iff $x - y \in \Lambda$. Let $M_\Lambda = \mathbb{R}^2/\Lambda$ be the topological space obtained as the quotient of $\mathbb{R}^2$ by this equivalence relation.

(a) Fix a lattice Λ and let $\pi\colon \mathbb{R}^2 \to M_\Lambda$ be the quotient map. Given $p \in \mathbb{R}^2$ and $r \in (0, \frac{1}{2})$, let $\hat{U}_p^r = B_r(p) \subset \mathbb{R}^2$, and show that $\pi|_{\hat{U}_p^r}$ is a bijection onto its image $U_p^r \subset M_\Lambda$. Let $\varphi_p^r = \pi|_{\hat{U}_p^r}^{-1}\colon U_p^r \to \hat{U}_p^r$. Let $r_0 = \frac{1}{2}\min\{|u| : u \in \Lambda, u \neq \mathbf{0}\}$. Show that the collection $\mathcal{A} = \{(U_p^r, \varphi_p^r) \mid p \in \mathbb{R}^2, 0 < r < r_0\}$ satisfies the conditions of the smooth manifold chart lemma, so M_Λ is a smooth manifold with smooth structure generated by this atlas.

(b) Let Λ_1 and Λ_2 be any two lattices in $\mathbb{R}^2$, and show that M_{Λ_1} and M_{Λ_2} are diffeomorphic (when equipped with the smooth structure from the previous part). **Hint:** *Start by finding a smooth map* $F\colon \mathbb{R}^2 \to \mathbb{R}^2$ *such that* $F(\Lambda_1) = \Lambda_2$.

(c) Let $\Lambda = \mathbb{Z}^2$ be the lattice generated by the standard basis vectors, and show that M_Λ is diffeomorphic to the torus $\mathbb{T}^2 = S^1 \times S^1$ with its standard smooth structure.

Remarks:

- The last two parts show that $M_\Lambda \approx \mathbb{T}^2$ for *any* lattice Λ. You should convince yourself that $\Lambda = \{(1,0), (0,1)\}$ corresponds to the square with opposite edges identified (the usual planar model of the torus), and that $\Lambda = \{(1,0), (\frac{1}{2}, \frac{\sqrt{3}}{2})\}$ corresponds to the hexagon with opposite edges identified (an alternate planar model). *(Question: what does "corresponds to" mean here?)*
- This works in any dimension: one can define a lattice in $\mathbb{R}^n$ as the integer span of a basis, and then get $\mathbb{T}^n = \mathbb{R}^n/\Lambda$.
- In two dimensions, one can also put some extra structure on M_Λ by identifying $\mathbb{R}^2$ with $\mathbb{C}$ and observing that the transition maps between charts in $\mathcal{A}$ are holomorphic. Thus M_Λ is an example of a **Riemann surface**, a manifold with (real) dimension two equipped with an atlas whose local coordinates are in $\mathbb{C}$ and whose transition maps are holomorphic. It turns out that while M_{Λ_1} and M_{Λ_2} are equivalent as smooth manifolds, they are not always equivalent as Riemann surfaces. Classifying tori up to equivalence as Riemann surfaces leads to something called *Teichmüller theory.*