Math 7350 — Geometry of Manifolds — Spring 2015

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HOMEWORK 3

Due in class *Monday, Feb. 16.*

1. *(Lee, Problem 3-2).* **Tangent space of a product manifold**
Let $M_1, \dots, M_k$ be smooth manifolds, and for each j, let $\pi_j \colon M_1 \times \cdots \times M_k \to M_j$ be the projection onto the M_j factor. Prove that for any point $p = (p_1, \dots, p_k) \in M_1 \times \cdots \times M_k$, the map

$$\begin{aligned}\alpha \colon T_p(M_1 \times \cdots \times M_k) &\to T_{p_1}M_1 \oplus \cdots \oplus T_{p_k}M_k \\ v &\mapsto (d(\pi_1)_p(v), \dots, d(\pi_k)_p(v))\end{aligned}$$

is an isomorphism.

2. *(Lee, Problem 3-4).* **A trivial tangent bundle**
Show that TS^1 is diffeomorphic to $S^1 \times \mathbb{R}$.

Remark: *You may find it interesting to consider whether or not the same is true for S^2. Be warned that we do not yet have the machinery in this course to answer this question.*

3. *(Lee, Problem 3-8).* **An alternate description of T_pM**
Let M be a smooth manifold and $p \in M$. Let $\mathcal{V}_pM$ denote the set of equivalence classes of smooth curves starting at p under the following equivalence relation: $\gamma_1 \sim \gamma_2$ if $(f \circ \gamma_1)'(0) = (f \circ \gamma_2)'(0)$ for every smooth real-valued function f defined in a neighbourhood of p. Show that the map $\Psi \colon \mathcal{V}_pM \to T_pM$ defined by $\Psi[\gamma] = \gamma'(0)$ is well defined and bijective.

4. **Quotient manifolds**
Let G be a group and E a smooth manifold. A *left action* of G on E is a map $G \times E \to E$, often written as $(g, p) \mapsto g.p$, that satisfies

$$\begin{aligned}g_1.(g_2.p) &= (g_1g_2).p \text{ for all } g_1, g_2 \in G, p \in E, \\ e.p &= p \text{ for all } p \in E.\end{aligned}$$

Suppose we are given a left action of G on E such that for every $g \in G$, the map $p \mapsto g.p$ is a smooth map from E to itself. This induces an equivalence relation on E by putting $x \sim y$ iff there is $g \in G$ such that $g.x = y$.

We say that the action is "free and proper" (see p.548–549 of Lee) if the following are true:

(i) for every $x \in E$ there is a neighbourhood U of x such that $g.U \cap U = \emptyset$ for every $g \neq e$;
(ii) for every $x, y \in E$ with $x \not\sim y$ there are neighbourhoods U of x and V of y such that $g.U \cap V = \emptyset$ for every $g \in G$.

Write $M = E/G$ for the quotient space of E by the relation $\sim$.

(a) Prove that if the action is free and proper, then M is a topological manifold.

(b) Let $\pi \colon E \to M$ be the quotient map. Show that π is a covering map. *(This means that for every $x \in M$ there is a neighbourhood $U \ni x$ such that $\pi^{-1}(U) = \bigsqcup_{\alpha \in A} V_\alpha$ for some disjoint open sets $V_\alpha \subset E$ such that $\pi|_{V_\alpha} \colon V_\alpha \to U$ is a homeomorphism.)*

(c) Prove that if E is a smooth manifold, then M has a unique smooth structure such that π is a smooth covering map. *("Smooth covering map" means that 'homeomorphism' is replaced by 'diffeomorphism' in the definition of covering map.)*